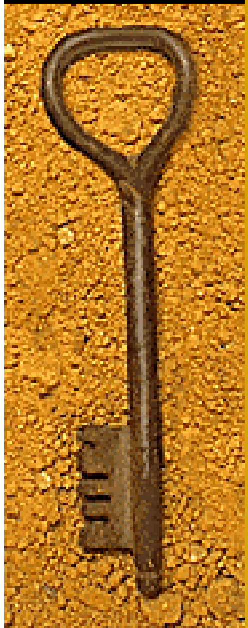

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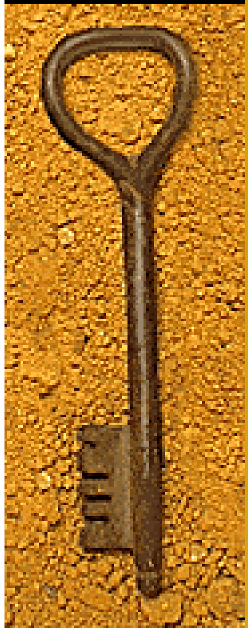
Inconsistent Mathematics

- Mathematics has traditionally been the hallmark of a science that proceeds by proof, and so is free of falsehoods and more so of inconsistency.
- Changing the basic logic used in mathematics to a paraconsistent logic makes mathematics in a weak sense paraconsistent: If there were to turn up some inconsistency in mathematics, it would not explode. But since there are no inconsistencies expected to arise there, a mathematician will not be inclined to forego the deductive power of standard First Order Logic.
- Changing set theory to a paraconsistent set theory makes mathematics paraconsistent in a stronger sense, since now the basic axioms are taken as the inconsistent axioms of naive set theory. There are now real inconsistencies – may be even inconsistent objects – in mathematics and the logic, therefore, has to be a paraconsistent one.
- And the inconsistency may not only reside with some elusive set theoretic entities, but there may be inconsistent numbers as well!



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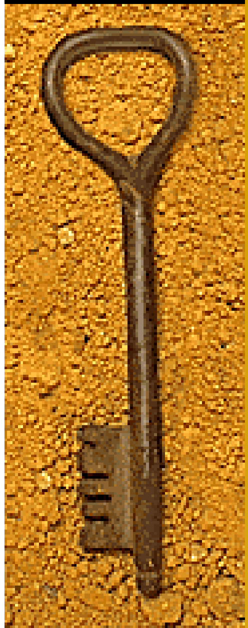
Inconsistent Mathematical Objects

- To have an inconsistent number theory means at least that within the theorems of number theory there is some sentence A with A being a theorem and ¬A being a theorem at the same time.
- Supposedly this corresponds to some object/number *o* being an inconsistent object.
- So inconsistent mathematics is connected to inconsistent ontology. Its underlying logic has to be paraconsistent.
- The problems with having F(a) and ¬F(a) for some object *a* seem not so pressing if *a* is some mathematical object than *a* being a physical object [see Chap. 17]: Mathematical objects are either non-existent – *mere* theory, taken instrumentally – or they are in some elusive Platonic realm where strange things may well happen.
- If on the other hand one is a reductionist realist about mathematics (mathematics being about structures of reality or mathematical entities rather being concrete entities dealt with by mereology) then inconsistent mathematics is as problematic as your cat being (wholly) black and not being (wholly) black at the same time.



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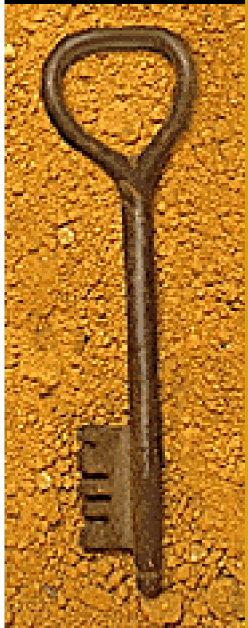
Inconsistent Mathematical Objects (II)

- Philosophers when concerned with mathematics focus on number theory, since the ontological questions of mathematics ("What and where are mathematical objects?", "Are there infinite sets?"...) and the epistemological questions of mathematics ("How do we know of numbers?", "Is mathematics merely conventional?"...) do arise already with number theory.
- Taking set and model theory as part of logic anyway, logicians are also mainly concerned with number, since a lot of meta-logical theorems make us of the device of arithemetization.
- The same goes for the general theory of automata and computability.
- We follow this focus here and so this chapter concerns itself mostly with arithmetic. Set theory as part of logic is deal with in Chap. 11.
- This may not be enough for a mathematician trying to assess the power of inconsistent mathematics. She looks for inconsistent theories at least of the power of the calculus. There are actually such theories, e.g. presented by Chris Mortensen [see *Further Reading*].



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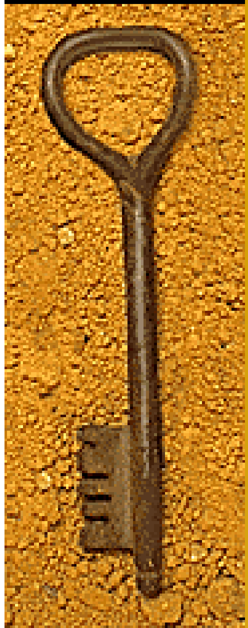
Non-Standard Arithmetic

- One of the most fundamental mathematical theories is arithmetic (as given for instance by the Peano axioms).
- Given its first order representation there are a lot of well-known theorems about arithmetic (e.g., arithmetic being incomplete).
- Given Compactness of **FOL** one can prove that there are non-standard models of arithmetic, which contain additional numbers over and above the natural numbers. These additional numbers behave consistently, however. Consistency provides them in the first place.
- Inconsistent arithmetic may concern itself with the opposite deviance: Having arithmetics where there are less numbers than in standard arithmetic.
- This is of outmost philosophical interest, since the infinite is a really problematic concept leading to the ever larger cardinalities of "Cantor's paradise", and finitism (in the sense of the assumption that there are only *finitely many* objects, even of mathematics) is therefore an option worth exploring and pursuing.



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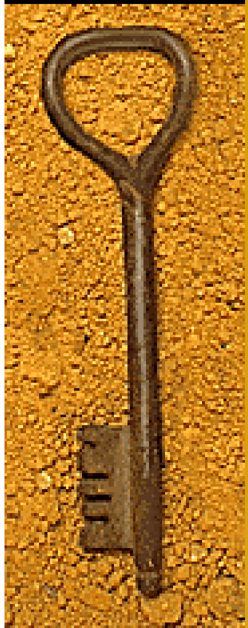
Arithmetic

- One has to distinguish a first order representation of arithmetic from a second order arithmetic. We are concerned mainly with first order representations.
- In distinction to an axiomatic arithmetic theory like Peano Arithmetic there is the arithmetic *N* (being the set of true first order arithmetic sentences in the standard interpretation).
- Many of the limitative theorems of meta-logic proceed by having a formal system that includes arithmetic and thus (by Gödelization or arithmetization) is able to represent its own syntax and thus recursive sets like the set containing pairs of proof numbers and numbers of the thus proven sentence.
- *N* is negation complete (either A or ¬A is in *N*), not axiomatisable, not decidable, and, of course, infinitely large.



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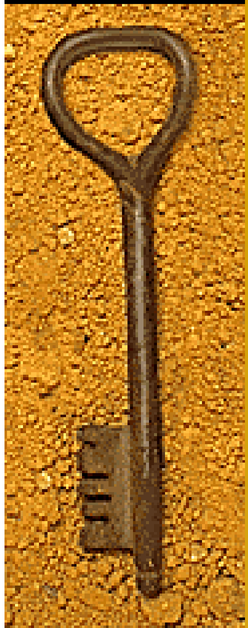
Arithmetic R#

- Robert Meyer was the first to give a non-triviality proof of a Relevant (paraconsistent) arithmetic.
- The system **R#** is an extension of the first oder version of Relevant logic **R** [see Chap. 4] with axioms mirroring those of Peano arithmetic save that the "⊃" in them has been replaced by the Relevant "→". Induction is present as a rule.
- Taking the logic **RM3** [see Chap. 9] gives us **RM3#**.
- One can extend that system with the ω-rule (to derive a generalization from infinitely many instances), this is **R##** or **RM3##**.
- **R#** is non-trivial in that 0=1 is not provable.
- This non-triviality can be established by finitistic methods.
- *Modus Ponens* for "⊃" fails in **R#**. If it was valid, Peano arithemetic would be a subsystem of **R#**.
- **RM3** or **R** may not be one's favourite logics, however, so we look here at inconsistent arithmetic in general, presupposing some basic paraconsistent logic like **LP**.



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Inconsistent Arithmetics

- Inconsistent Arithemtics that are finite may have any finite size you like. They contain one largest number. Since we do not know which number really is the largest we may assume that one of these arithmetics is true, although we don't know which. Which one it is is not that important, since all these arithmetics have common properties:
- Let n be some natural number, then let N_n be a set of arithmetic sentences. These sets have the following properties [cf. (Priest 1994)]:

  (i) $N \subset N_n$.

  (ii) N_n is inconsistent.

  (iii) $A \in N_n$ for a (negated) equation A concerning numbers < n if and only if $A \in N$.

  (iv) N_n is decidable.

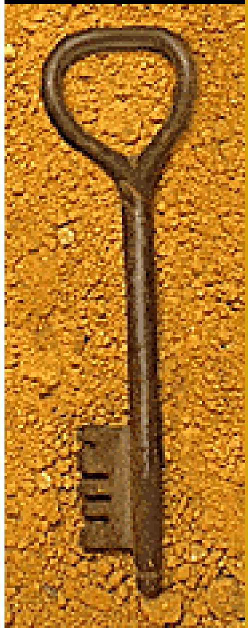
  (v) N_n is representable in N_n (thus we have a N_n truth predicate).

  (vi) For the proof predicate B( ) of N_n every instance of

  $$B("A") \supset A \qquad \text{is in } N_n.$$

  (vii) If A is not a theorem of N_n $\neg B("A") \in N_n$.

  (viii) For the Gödel sentence G for N_n $G \in N_n$ and $\neg G \in N_n$.



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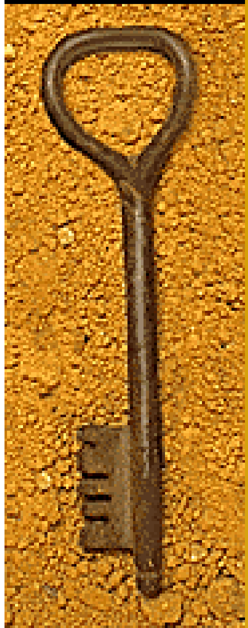
Inconsistent Arithmetics (II)

- A inconsistent arithmetic N_n thus has quite remarkable properties:

  - by (i) we have that it is complete, since *N* is.
  - by (ii) and (viii) we have, of course, that it is inconsistent.
  - by (iv) it has all the nice properties that *N* does not have, although N_n is complete!
  - by (v) we can in the language of arithmetic define a truth predicate for that very same language.
  - by (vi) N_n has an ordinary proof predicate.
  - by (vii) in conjunction with (iii) we have not only that N_n is not trivial (by excluding some the equations that are excluded by *N*), but that this non-triviality can be established within N_n itself.

- How do we get this?



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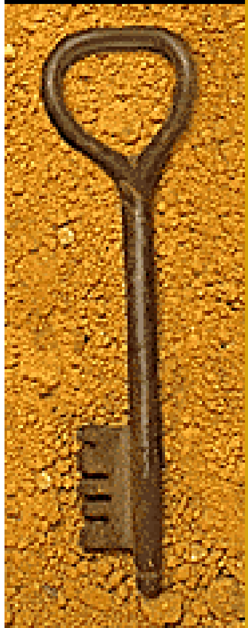
Inconsistent Arithmetics (III)

- *Proof Outline*:
  - A theory with less numbers than *N* can have less counterexample to a given arithmetic sentence. Thus it contains at most more sentences (as true). This holds in general (called "Collapsing Lemma"). Therefore (i). So we do not lose any of the power of *N* by switching to N_n.
  - Since *N* is negation complete *adding* any sentence (as true) means adding a sentence for which the negation is already in *N*. Thus the resulting theory contains for at least one A, A and ¬A. Thus (ii). This means that the logic of these arithmetic theories has to be a paraconsistent logic.
  - Representability of truth is a consequence of (iv) and (i). The same holds for the representability of the proof predicate, (vi). Once the proof predicate is representable in the decidable theory N_n we can represent non-provability, and thus have (vii) and finally (viii).
  - (iii) is the most interesting property and results from the way the domain of a corresponding model is constructed:



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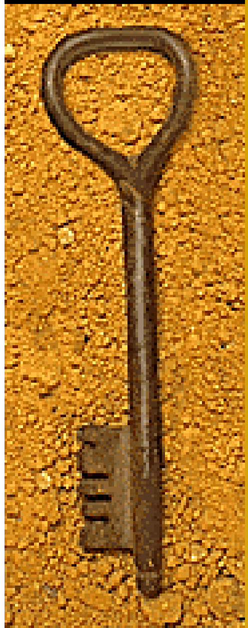
Inconsistent Arithmetics (IV)

- A model of a theory N_n is constructed as a filtering of an ordinary arithmetic model.
- In general one can reduce the cardinality of some domain by substituting for the objects equivalence classes given some equivalence relation (i.e. instead of objects o_1, o_2 ... we have $[o_1]$, $[o_2]$...). The equivalence classes provide then the substitute objects. Since the objects within the equivalence class are equivalent in the sense of interest in the given context the predicates still apply (now to the substitute object).
- The trick in case of N_n is to chose the filtering which puts every number $< n$ into its equivalence class, and nothing else; and puts all numbers $\geq n$ into n's equivalence class.
- As a result of this for $x < n$ the standard equations are true (of $[x]$), while in case of $y \geq n$ *everything* that could be said of such a y is true of $[n]$. So we have immediately $n = n$ (by identity) and $n = n + 1$ (since for $y = n + 1$ in N this is true).



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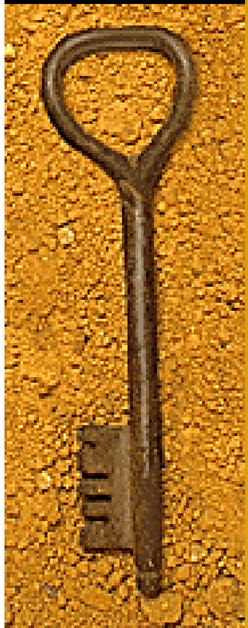
Inconsistent Arithmetics (V)

- The domain of a theory N_n so is of cardinality n.
- n now is an inconsistent object of N_n.
- If for the moment we picture the successor function by arrows we can picture the structure of a model of N_n thus:

$$0 \rightarrow 1 \rightarrow \dots \rightarrow n \rightarrow$$
$$\uparrow \quad \downarrow$$
$$\leftarrow$$

Such models are called "heap models".

- The logic modelling N_n has to be paraconsistent. And is has to have restrictions on standard first oder reasoning as well:



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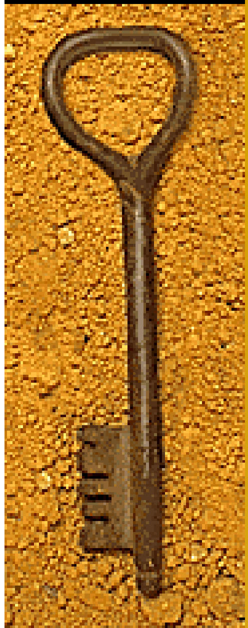
Inconsistent Arithmetics (VI)

- (=E) *cannot hold* for n of N_n if triviality is to be avoided:

1. n – n = n – n	Theorem (for any number)
2. n = n + 1	Assuming n to be the largest number
3. n – n = n – (n+1)	(=E) 1, 2
4. n – n = 0	Theorem
5. 0 = n – (n+1)	(=E) 4, 3
6. 0 = (n+1) – n	Commutativity, 5
7. ((n+1)-n)×(b-a)/((n+1)-n)=((n+1)-n)×(b-a)/((n+1)-n)	Theorem
8. (0×(b-a)/((n+1)-n) = ((n+1)-n)×(b-a)/((n+1)-n)	(=E) 7, 6
9. 0×(b-a)/((n+1)-n) = 0	Theorem
10. 0 = ((n+1)-n)×(b-a)/((n+1)-n)	(=E) 8, 9
11. ((n+1)-n)×(b-a)/((n+1)-n) = b – a	Theorem
12. 0 = b – a	(=E) 10, 11
… 18. a = b	by some Theorems for “+”, “–“ ■

- A logic with unrestricted (=E) so yields a=b for *any* numbers!
  [Remember the restriction of (=E) in Chap. 6; see **UL4** in Chap. 20]



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Inconsistent Arithmetics (VII)

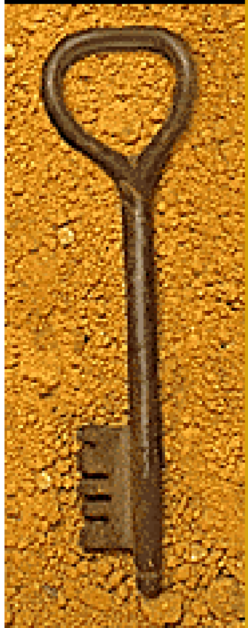
- Mortensen choses **RM3#** as basic system and finitizes it by substituting for a number n the number n *modulo* some m. Thus the domain becomes {0, 1, 2, ... m-1}
- The resulting arithmetic **RM3$^{\mathbf{m}}$** is complete, non-trivial and decidable.
- **RM3$^{\mathbf{m}}$** is *axiomatisable* by adding to **RM3#** the axioms:

$$\vdash 0 = m$$

and all instances of the following axiom scheme for $n \in \{0, 1, ... m\text{-}1\}$:

$$\vdash (0 = n \leftrightarrow 0 = 1).$$

- The approach "*modulo* some m" has at least the same deviant results than the heap models mentioned before: In **RM3$^{\mathbf{5}}$** we have $4 + 2 = 6$ (since **RM3$^{\mathbf{5}}$** is complete, i.e. has all theorems of *N*) and $4 \times 6 = 4$ (since "6" denotes 1). And the approach "*modulo* some m" has these deviant sentences for some *known* numbers!



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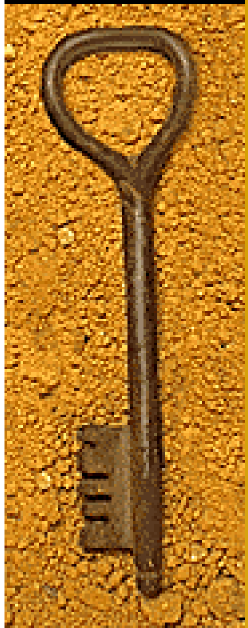
Finitization

- Arithmetic is constructed thus as a finite theory. One can generalized the steps of this procedure to apply it to other mathematical theories.
- Van Bendegem distinguishes the following steps:
  (i) Take any first-order theory T with finitely many predicates. Let M be a model of T.
  (ii) Reformulate the semantics of T in a paraconsistenf fashion (i.e. the mapping to truth values and overlapping extensions of P^+ and P^-).
  (iii) If the models of M are infinite, define an equivalence relation R over the domain D of M such that D/R is finite.
  (iv) The model M/R is a finite paraconsistent model of the given first-order theory T such that validity is at least preserved.
- The restriction to theories with finitely many predicates is no real restriction in any field of applied mathematics or formal linguistics, since no physical device (be it human or machine) can store a non-enumerable list of basic predicates.

- [Van Bendegem then hints at finite version of the theory of integers and the theory of rational numbers.]



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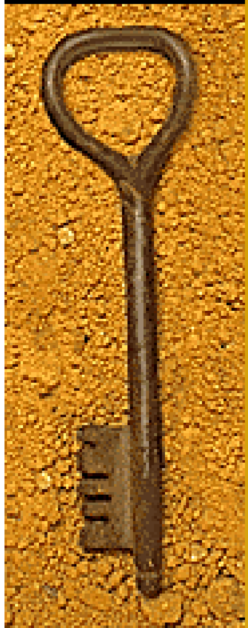
Paraconsistent Löwenheim/Skolem

- The *Löwenheim/Skolem-Theorem* is one of the limitative or negative meta-theorems of standard arithmetic and **FOL**. It says that any theory presented in **FOL** has a *denumerable* model. This is strange, since there are first order representations not only of real number theory (the real numbers being presented there as uncountable), but of set theory itself. Thus the denumerable models are deviant models (usually Herbrand models of self-representation), but they cannot be excluded. [They can be excluded in some 2nd order semantics for 2nd order set and number theory, see (Shapiro 1991).]
- Given the general procedure to finitize an existing mathematical first order theory using paraconsistent semantics, there is a paraconsistent strengthened version of the *Löwenheim/Skolem-Theorem*:

> Any mathematical theory presented in first order logic has a *finite* paraconsistent model.



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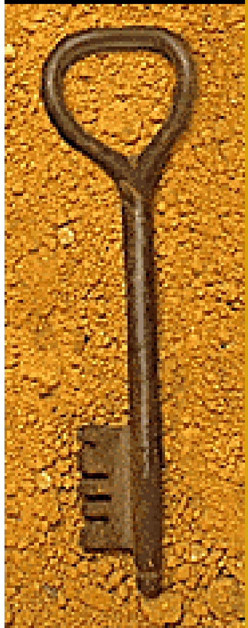
Finite Inconsistent Arithmetics – Assessment

- A mathematics that does not committ us to the infinite is a nice thing for anyone with reductionist and/or realist leanings.
- As far as we know the universe is finite, and if space-time is (quantum) discrete there isn't even an infinity of space-time points.
- The largest number may be indefinitely large. So we never get to it (e.g. given our limited resources to produce numerals by writing strokes).
- If there is a largest number n there is the corresponding inconsistent arithmetic N_n. We can presuppose N_n being our arithmetic. Since N and N_n agree on all finite and computational mathematics it is hard to see whether we lose anything important at all by switching to N_n.
- Since we have paraconsistency anyway for other reasons, we get this finitism *for free*, it seems. So why not take it? In as much as N_n is correct no correct reasoning trancends the finite. Hilbert wouldn't have rejoiced, probably, since N_n of course is inconsistent itself.
- The drawback of all this is, of course, the problem of an ontology of inconsistent entities – at least if you are a realist. [see Chap. 17]



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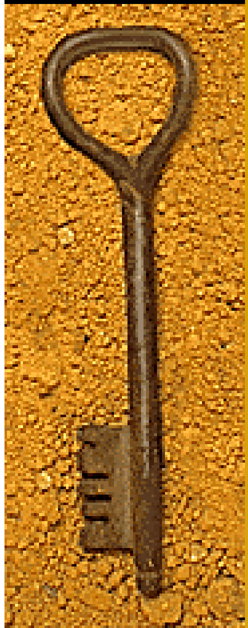
LFI-Based Inconsistent Arithmetic

- Benassi and Gentilini (2003) have developed a paraconsistent provability logic based on a paraconsistent arithmetic that is an extension of a sequent version of the LFI-system **bC** [see Chap. 8].
- The system **PCA** consists of axioms:
  - defining the primitive recursive functions,
  - for "=" (i.e. reflexivity and functionality of "=", as well as substitutivity of identicals within primitive recursive functions),
  - defining each numeral as the successor of another, starting with "0";

  these axioms are supplemented by a rule of induction. By this **PCA** is recursively axiomatized.
- Given that standard Primitive Recursive Arithmetic, **PRA**, is non-trivial and negation consistent, so is **PCA**, since any of its proofs is available in **PRA**. For atomic formulas the systems are even identical.
- This means that m=n is not provable in **PCA** for different numerals. Thus it is *not* a finite arithmetic like the ones we dealt with before. For one and the same numeral m, however, we can have without trivialization: $\emptyset ||\!\!- (m = m \wedge m \neq m)$

  ["||-" being the sequent symbol again.]



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LFI-Based Inconsistent Arithmetic (II)

- That

  $\vdash_{PCA} \emptyset ||\!\!-(m = m \wedge m \neq m)$

  does not trivialize the system means that **PCA** allows for objects/numbers that are (also) *not identical* to themselves. The presense of the contradiction with respect to m=m does not lead to triviality, exhibiting the paraconsistent character of **PCA**.
- If P( ) is any arbitrary open formula and á, é any closed terms that represent numbers, then **PCA** does not have substitution of identicals, i.e.

  $\not\vdash_{PCA} \emptyset ||\!\!-(\text{á}=\text{é} \supset (P(\text{á}) \equiv P(\text{é})))$

  neither does this hold in sequent form.
- **PCA** as a theory shows the pecularities of **bC** [cf. Chap. 8]**,** it is interesting for us since Benassi and Gentilini use it to introduce a provability predicate the properties of which can be compared to a standard provability predicate [see Chap. 13]. A provability predicate for **PCA** can be expressed within the system itself, because it contains enough of the primitive recursive functions and so – by arithmetization of its syntax – can represent the provability relation within itself.

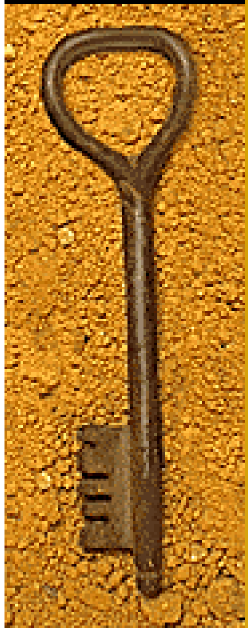
Further Perspectives

- If there are inconsistent versions of more elaborated mathematical fields like the calculus one may draw some general philosophical conclusions:

  (i) If there are corresponding inconsistent versions of these mathematical theoris with comparable strength to the original theories then consistency is not the fundamental mathematical concept, but functionality (of the respective basic concepts) may well be. As Mortensen sums up:

  > [T]he fact that the same functional structure can underlie inconsistent, incomplete, or classical theories suggests that the functional aspects of mathematics are more important than squabbles at the sentential level over [Explosion], inconsistency, incompleteness, *etc*.

  (ii) If the justification of mathematics depends on its applicability and the inconsistent versions are of comparable applicability then they are justified not just as mathematical theories, but even in the wider perspective of grasping fundamental structures of reality; there no longer will be available the argument from mathematical describability to the consistency of the world.



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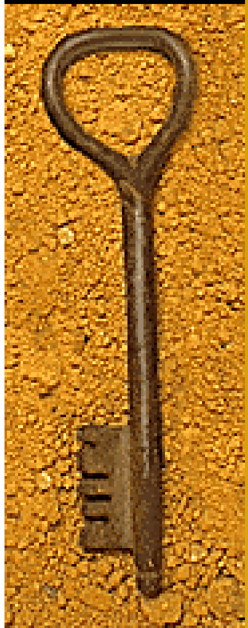
Questions

- (Q1) N_n is decidable – although the procedure maybe quite involved and beyond our limited physical resources. Why? Why is it also finitely axiomatisable?

- (Q2) Given that provability is representable in N_n *and* given that N_n is decidable, *and* given a representation of truth, why is the Gödel sentence for N_n

$$(\gamma)\ \neg B(\gamma)$$

provable, as well as its negation?



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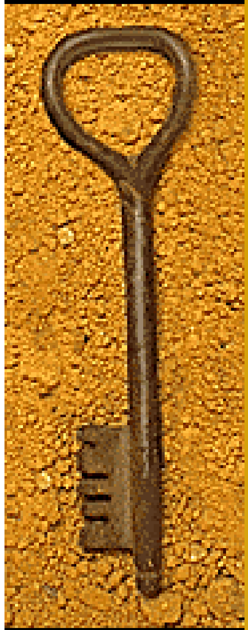
Excercises

- (Ex1)
- (Ex2)



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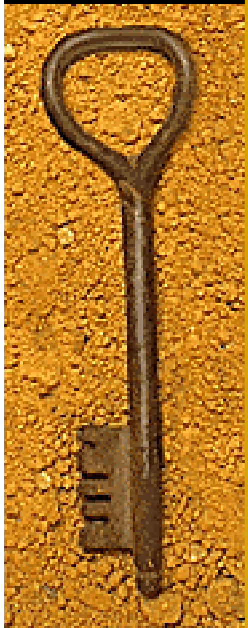

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Further Reading

- The basic – and only – monograph on inconsistent mathematics is: Mortensen, Chris. *Inconsistent Mathematics*. Dordrecht/Boston/ London, 1995.
- For R# see: Meyer, Robert. "Relevant Arithmetic", *Bulletin of the Section of Logic of the Polish Academy of Science*, 1976, pp. 133-37.
- Graham Priest wrote a couple of papers dealing with inconsistent number theory: (Priest 1994, 1994a, 1996a, 1997).
- On the use of arithmetic in meta-logic and the corresponding results see (Boolos/Jeffrey 1989).
- On the infinite in general, large cardinalities and their role and history in set theory see (Lavine 1998) and (Moore 1990).
- A statement of paraconsistent finitism is: van Bendegem, Jean Paul. "Strict, Yet Rich Finitism", in:Wolkowski, Z. (Ed.) *First International Symposium on Gödel's Theorems*. Singapore, 1993, pp. 61-79.