Universality in Set Theories

A Study in Formal Ontology

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Any mathematical consideration must be founded on the notion of ‘allness’ or ‘quantification’ as a basic category of logic which cannot be subject to further analysis whatsoever.

(Ernst Zermelo)

What surpasses all that is finite and transfinite is no “Genus”; it is the single, completely individual unity in which everything is included, which includes the “Absolute”, incomprehensible to the human understanding.

(Georg Cantor)

Content

INTRODUCTION

This essay discusses the fate of universality and a universal set in several set theories. It presupposes a general background in logic and general knowledge of set theoretic basics. Even basic points are repeated if the context of discussion profits from a short reminder, but no systematic survey of the different systems is attempted. The book aims at a *philosophical* study of ontological and conceptual questions around set theory. A *formal* exposition of some consistent set theories with a universal set and related theorems can be found in (Forster 1992) and the sources mentioned there. A comprehensive formal exposition of paraconsistent set theories does not yet exist; some sources are mentioned in chapter V.

Set theories *are* ontologies. They posit entities and claim that these exhibit some essential properties laid down in the set theoretical axioms. Like Zermelo (in the opening paragraphs of Zermelo 1908) Fraenkel, in his early introduction to set theory (1919/1928^2) explicitly outlines this *axiomatic approach*:

> According to the essence of this method we refrain to *define* the concept of set or to analyze it, we rather start with some axioms in which the concept of set like the relation 'to be contained as an element' occurs, and in which the existence of some sets is postulated. The concept of set is implicitly established by the totality of these axioms.

Collecting these postulated entities poses the problem of universality. Is the *collection* of the set theoretical entities itself a set theoretical entity? What does it mean if it is, and what does it mean if it is not? To answer these questions involves developing a theory of the *universal set*. For a start we may define the universal set as U = {x | x = x}.[1] As set theories extend first order logic with identity (**FOL**) or some variant of it (in a non-standard logic) they contain the axiom: (∀x)(x = x). U thus comprises the *whole domain* of the language. Tautologically whatever exists exists. So, supposedly, *there are* all existents. Why not continue: So *there is* the totality of these existents? Why shouldn't they be collectible? After all, set theoretical quantification runs over *all sets*, doesn't it? If, however, that totality was an additional entity we could collect another totality including it – and so forth, it seems. Unless this totality possessed a nature *sui generis*, setting it apart from ordinary things and sets.

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1 Standard symbols are used. See the appendix on notation.

Some of the set theories are pure set theories: their domain consists entirely of sets, all variables range over sets. Some of the theories contain proper classes in addition to sets (improper classes). Some theories contain additional ‘urelements’ (i.e. objects which are neither sets nor classes, but something to be collected into sets or classes). Some other theories use numbers as basic entities, not reduced to set theoretic construction. The discussion here will mention these differences, but will not use a neutral formalization, which applies to all theories; this would require using one type of variables and sortal predicates like “set”, “ordinal” etc. to restrict quantification to the appropriate type, e.g. “(∀x,y)(Set(x)∧Set(y) ⊃ …”. The drawback of this formalization would be its contrast to the respective textbooks and articles. Additionally it would be very cumbersome, e.g. having all the sethood statements in pure set theories like **Z**, the axiomatization of which would include now a new axiom “(∀x)Set(x)”, which had to be used all over to get rid of the sethood requirements in the antecedents of statements. Thus when discussing mixed type systems sortal predicates may be used, but not with pure set theories, and not with systems which only distinguish sets from (proper) classes; in the latter case lower case variables refer to sets, upper case variables to (proper) classes.

The existence of urelements is important for the broader ontological picture [cf. Chap. VI], but the presence or absence of a basic set of urelements does not change the treatment of universality in many set theories (like **ZFC**). One may ask oneself what sort of things might be chosen as urelements. If physical entities are chosen, there are – in the light of our best physical theories – only finitely many of them, which can be collected into a set of urelements. Physical objects may have their own principles of composition (like mereology). Their presence does not influence the question whether all sets can be collected into a universal set. Once sets are present, there seems to be no need for further elementary logical (abstract) entities like numbers. A collection of urelements that matches the sets in cardinality seems highly dubious, as one may suppose every urelements to have a singleton and any two of them to be elements of their pair set – etc. In some cases (like *Specker’s Theorem* [in Chap. IV]) we have to talk about urelements.

Several issues related to set theory will not be discussed here: We are not much concerned with the *epistemology* of mathematics in general or set theory in particular. With respect to epistemology all the theories discussed here are *prima facie* in the same boat. We may, however, raise some questions concerning whether understanding universality raises *additional* epistemological problems. We start with the ordinary working assumption that we have some concepts and ideas of sets and numbers and set theory tries to systematize them. Therefore we will not be concerned with the general issue of abstractness or 'Platonism' either. Again, *prima facie*, all the theories discussed here are in the same boat. We may raise some questions concerning whether some ways of understanding universality or the set theoretic universe raise *additional* ontological problems.

For the set theoretical anti-realist our study is just a case study in formal ontology and its models. Nonetheless some such ontologies might be more useful than others even if all are – strictly speaking – false, as there are no sets whatsoever. Even if there are no sets some set ontologies may be more helpful fictions than others. They help in systematizing mathematics, which again, even if without subject matter itself, helps as part of science in describing reality. Even Russell held at times that sets are just a manner of speaking, but not part of the furniture of the world (cf. Russell 1914). Nonetheless as they correspond, for Russell, to the fundamental 'propositional functions' talk about them is neither arbitrary nor idle. One theory is singled out as capturing or founding *our* mathematics.[2]

For the set theoretical realist one set theory might be better in capturing set theoretical reality than another. Either one has to assume U or one has to assume that U does not exist. Our study then is one attempt to ascertain which option we have to take. Even if one endorses 'plenitudinous Platonism' (the thesis that *all* consistent mathematical theories correspond to some part of the realm of abstract entities[3]) the issue of U is not idle. The different set theories might then be taken to deal with different areas of abstract entities. In one area there might be something like U in the other area not. Nonetheless, one may argue that one of the areas has more right to be considered as making up *sets* as we have an intuitive notion of SET, which may be explicated better in one theory than in another. Even if there are several areas of abstract entities which are set-like one area may be the *intended standard model* corresponding to our concept of SET. In this perspective our study is concerned with the *conceptual issue* of analysing our concept of sets. Comparing the different theories and weighing the advantages and disadvantages of incorporating U into a set theory (i.e. the gain and the strain of related theorems/facts in relation to our intuitive

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[2] Ironically Russell demanded at that time, of course inspired by the antinomy of the set of non self-membered sets, that saying either that a set is a self-member or that it is not should be *meaningless* (not just false). A requirement which excludes the Axiom of Foundation. Naturally Russell thought in terms of his theory of types, which by definition fulfils this requirement, but, nevertheless, was laying down *conceptual constraints* on a feasible concept of set.

[3] Cf. Balaguer 1998. The view that consistency proves sufficient to take a mathematical theory seriously has many variants. These include Hilbert's formalism and fictionalism, a theory of course denying the existence of abstract entities. Included is as well Cantor's 'theological Platonism', which has all consistent transfinite entities existing as ideas in God's mind.

understanding of sets) we may come to a result whether our intuitive concept SET involves the assumption of a universal set or rather some other picture. Even if set theory was not *of* sets in the referential sense (as there might be no sets at all) set theory would be *of* sets in the intensional sense of setting out our conceptions of sets. And our question here is whether the universal set crucially belongs to these conceptions, or whether it is an idea at the periphery of these conceptions, only to be rejected on second thoughts about its consequences.

ZF asserts that some collections we have naively thought of as *sets* (the set of ordinals, cardinals, the universal set) are not sets – i.e. they do not exist for **ZF** *itself*. Most surprising is this claim for U, as {x| x = x} seems so natural. That our untutored intuitions have to be partly corrected at the foundation of science, however, occurs not just in **ZF** but – one may well argue – at the foundations of physics (e.g. with our untutored intuitions about the locality of particles or the properties of time) or in biology (e.g. with the changeability of some organisms' essence/species). Thus the mere correction of our prior, untutored understanding of a basic concept does not establish *in itself* that **ZF** goes wrong. The argument has to concern whether this is the best option, what repercussions this step has, and whether the resulting concept of sets provides a more coherent (unified and comprehensive) understanding of sets.

The antinomies of semantics and set theory have to be treated somehow to provide a coherent systematic account of the notions involved. The same applies to the presupposed concept UNIVERSALITY in set theory. One can well do in large parts without treating these problems. Many a textbook works with informal set theory. They miss then, however, a comprehensive account of sets. What their success – *inter alia* (compare similar arguments in semantics) – shows is that the problems occur placed within an otherwise viable world view or viable procedures in semantics or set theory, say some version of semantic realism or of constructive representationalism. I, therefore, neglect theories that argue from antinomies and universality to some form of mysticism, ineffability, anti-representationalism, or what not.[4]

The following issues set the theme for much of the discussion here:

1. How can one avoid slipping into a 'theory' that universality is ineffable?

[4] Patterson (2008) extends his anti-representational program to mathematics. Postmodern authors endorse Wittgenstein's *Tractarian* mysticism about ineffability.

2. Are there different aspects to universality in set theory, which stand in conflict to each other?
3. What aspects of universality are embedded within our concept SET?
4. May inconsistency be the price to pay to circumvent ineffability?

and most importantly:

5. How far can axiomatic ontology take us in postulating our way out of the problems around universality?

Chapter I starts with the treatment of universality in standard set theory **ZF**. This raises issue (2): universal comprehension and universal collection (into a domain of all sets) seem to be incompatible. The approaches considered in chapters II – V raise issue (5) of axiomatic ontology. As already in chapter I the spectre of ineffability, issue (1), raises its head. Some systems engender their own incompatibilities between aspects of universality (like the tension between the universal collection into a domain *vs*. universal possession of a singleton), issue (2) again. And we have to ask which of the systems have a claim to be more 'natural' or 'intuitive' for us, issue (3). Chapter I also articulates one picture of universality: the iterative hierarchy. Articulating this picture raises issues (1), (3) and (5). Chapters II asks whether the problems can be avoided by moving either to Second Order Logic (**SOL**) or to an abstract realm 'broader' than the one of set theory, category theory for instance. Issue (4) is confronted in chapters V and the wider ontological discussion of 'noneism' in chapter VI. Chapter VI tries to come to grips with the methodological problems around issues (3) and (5).

Although this is a systematic study (i.e. not an historical investigation into the development of several set theories) sometimes it may be illuminating to mention and consider side-remarks made by their foundational authors. In these remarks one can at times discern the broader ontological picture the author works with.[5]

[5] As this is no detailed historical study I often omit giving the detailed source of a side-remark, but refer the interested reader to the comprehensive studies by Ebbinghaus (2007) on Zermelo, Dauben (1979) on Cantor, as well as (Lavine 1998). Some remarks have entered the set theoretical folklore and can be found in many introductory books (e.g. Potter 2004) or (Deiser 2010), which contains many quotes of the founding fathers, following the development of set theory.

I

ITERATION, FOUNDATION, AND REFLECTION

ZFC has become in its first order axiomatization the accepted set theoretic standard. We take, as usually done, as **Z** the system containing the Axioms of Extensionality, Pairing, Powerset, Sums, Separation and Infinity. **ZF** adds Foundation and Replacement, like Separation an axiom schema. **ZFC** adds the Axiom of Choice.[6] **ZFC+GCH** adds the Generalized Continuum Hypothesis to **ZFC** – and so on for stronger axioms.

The antinomies (like *Russell's Paradox*[7]) are often taken as showing that Naïve Comprehension

(NC_1) $(\exists y)(\forall x)(x\in y \equiv \varphi(x))$

(NC_2) $(\forall F)(\exists y)(\forall x)(x\in y \equiv F(x))$

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6 Historically this is misleading as Zermelo included the Axiom of Choice in his system, where he used it to prove well-ordering (in 1908). He also has an extra axiom for the empty set, ∅, but as in **FOL** the domain cannot be empty, one does not need this axiom, but gets ∅ by separation. In the 1920s Fraenkel and von Neumann and Skolem added Replacement. Zermelo's original system did not contain Foundation, but his system of 1930 does. His 1930 system **ZF'** leaves out the Axiom of Infinity as he then considered it to be an extra-logical existence assumption. Zermelo's formulation was not confined to **FOL**, but Skolem's clarification of 'definite' property as used in an instance of Separation led to first order **ZFC**. Cantor already stated and used both the Axiom of Choice and Replacement.

7 'Antinomy' will be used for a contradiction provable given some theory and its logic. A 'paradox' is just a theorem contrary to our expectations and prejudices. Already Zermelo stressed the importance of this distinction, as otherwise one sees the likes of too many antinomies where there are only paradoxes. Unfortunately usage is not so clear nowadays. By the way: The antinomy unfortunately called 'Russell's Paradox' was discovered some years earlier by Zermelo. It leads back – as many antinomies – to negative self-application of a property/predicate, the idea behind the canonical proof of *Cantor's Theorem*, which served as the context of discovering 'Russell's Paradox'.

is wrong. The assumption that every concept/property[8] has an extension, which is a set, is considered rejected. The first order axiom schema (NC_1) or the second order axiom (NC_2) are sometimes called 'naïve set theory'. They were by no means present in all approaches to set theory introduced in the 19th century. Cantor's original set theory was concerned with combinatorial multiplicities. At times, though, he considered sets as 'united by a rule', which sounds like Comprehension. Comprehension was certainly present in the logicist approach to set theory of Frege and Russell.

Now, take a version of Comprehension: the Russell Set, defined as $R = \{x \mid x \notin x\}$, and taking '$x \notin x$' as the open formula $\varphi(x)$ or the property F yields the famous antinomy: $R \in R \wedge R \notin R$. The defining property of NOT BEING A SELF-MEMBER seems to violate the constitutive assumption behind Naïve Comprehension by not having an extension, on pains of inconsistency.

There is another reading of *Russell's Paradox*, however. Proceeding to Zermelo's *Aussonderungsaxiom* (Axiom of Separation)[9] or not-naïve Comprehension scheme (of set theory **Z**)

(AS) $(\forall x)(\exists y)(\forall w)(w \in y \equiv w \in x \wedge \varphi(x))$

the property NOT BEING A SELF-MEMBER can be used to derive:

(NU) $\neg(\exists x)(\forall y)(y \in x)$

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8 In the context of this essay I take "concept" and "property" to be synonymous within set theories, as is usually done. In (natural language) semantics concepts may be said to refer to properties, which are often not taken as sets. Set theoretic ontology is less fine grained. A distinction is made between formulas expressing a concept/property and the concept/property. CAPITALIZATION is used to signal a concept/property. Reflecting *on* set theory and its relation to our cognition concepts (like the concept SET) are taken in their usual sense as cognitive, and whether they are captured and explicated by a theory (say, of 'sets') is the matter of debate.

9 To be precise: It is a schema in the wff φ. Any set can be separated by *this* axiom schema which corresponds to a wff in the language of the theory. The constructible universe L (used in Gödel's relative consistency proof for the Axiom of Choice and the Continuum Hypothesis) consists only of such sets, which requires restricting the powerset operation to constructible subsets.

the denial of a universal set.[10] What *Russell's Paradox* shows on this reading is that the assumption of the existence of a universal set is illicit. *Cantor's Theorem* establishes that the powerset $\wp(x)$ of a set *x* has a larger cardinality than *x*. Cantor's classical proof refutes the supposition of a bijection *f* between *x* and $\wp(x)$ by considering the subset $\{x | x \notin f(x)\}$. If *x* is the universal set this naturally introduces the Russell Set (being an element and a subset of the universal set). The idea of a universal set thus stands in tension to a core ingredient of the concept SET: that every set has subsets, which should be collectible. "$\subseteq$" is as central to set theory as "$\in$": one of them provides a sufficient foundation:

(D$\subseteq$1) $x \subseteq y \overset{\text{def}}{=} (\forall z)(z \in x \supset z \in y)$

(D$\in$1) $x \in y \overset{\text{def}}{=} \{x\} \subseteq y$

The Powerset Axiom focussing on "$\subseteq$", therefore, deserves a special role in any set theory, as Comprehension and/or Separation focus on "$\in$". That $U \in U$ may seem less unnatural than $\wp(U) \subseteq U$ and $\wp(U) \in U$.[11]

Comprehension is fine as long as we restrict the domain of objects to be comprehended. If we assume that there is no universal set or domain even Naïve Comprehension need not lead to the antinomies, as one cannot take for granted that R (or a similar cause of trouble) belongs to the objects (sets) to be comprehended. (AS) provides the safe formalization of this idea. The property NOT BEING A SELF-MEMBER can be taken as having an extension now that (AS) has been adopted. Any property has an extension *relative* to a base set. And if *a* is the base set for an instance of (AS) with '$x \notin x$', the extension of the subset corresponding to NOT BEING A SELF-

[10] *Proof* (Outline). Assume U exists. Take U as the base set *x* in (AS). The first conjunct on the right side of the biconditional can then be eliminated, being logically true. One arrives at the form of (NC_1) and the usual reasoning to the Russell Paradox goes through. Reject the existence assumption concerning U by arriving at the contradiction. ■ This proof can already be carried out in a weak subsystem of **Z**, like Kripke-Platek set theory **KP**. Membership can hardly be indeterminate for a set theoretic realist. Even if this had some plausibility for some sets, with respect to U something is in the universe or is not. Avoiding the *Russell Paradox* by banning R from U leaves us with the mystery where to put R then, or with the option that some collections cannot be sets, which leads to a set/non-sets distinction, we will look at in chapter II. Indeterminate membership plays no role here; theories without *tertium non datur* will be considered in chapter V, but giving up *tertium non datur* may mean rejecting *both* $R \in R$ and $R \notin R$.

[11] Even the problem with Frege's 'basic law' (V) goes back to this, since Frege at the same time defines extensions as objects (i.e. first order entities) and puts them in basic law (V) in correspondence to courses of values (predication) of concepts (i.e. second order entities), by *Cantor's Theorem* there have to be more extensions of concepts (namely sets of objects) than objects (cf. Boolos 1998, pp.135-54). Because of the complete absence of a Powerset Axiom we do not consider set theories like **KP** (Kripke/Platek set theory) in detail.

MEMBER relative to *a* is *a* itself (as by the Foundation Axiom no set is a member of itself, so that all members of *a* satisfy the condition $x \notin x$).

The discovery behind the set theoretical antinomies then consists not in a claim about properties

(NNC) Not every property has an extension.

but in a claim about universality

(NU') There is no universal set.

Both claims are ontologically substantial and surprising. Hilbert, for instance, thought that conception formation was in trouble, as the idea that being able to determine whether something falls under a concept does not suffice for the concept's existence.

The argument against U works with Separation. Using (NC) leads to the antinomy. One reading of the antinomic argument can also be that it uses the assumption that the Russell Set *R* is part of 'all' objects (i.e. *within the range* of "∀"). The range of "∀" on pains of contradiction thus cannot be universal, *R* lying outside of it. Thus there is no unrestricted quantification over all collections. If "∀" ranges over all sets, *R* cannot be a set after all. The collection of non self-membered sets turns out *to be* the range of "∀" in **Z** because of the Axiom of Foundation (i.e. turns out to be the iterative hierarchy V itself)! In this reading of the antinomic argument again *a set* of all sets is excluded. The reasoning poses two problems we come back to again and again: (i) (NC) still allows building the forbidden collections U and *R*, and (ii) the reasoning invites our naïve bewilderment *where* some collection is 'to be' when outside of the range of "∀".

The naturalness of the idea of universality or a universal set may be related to the *Calculus of Classes* (cf. e.g. Hilbert/Ackermann 1928, Chap. 2).[12] Textbooks unhesitatingly speak of a 'universal class' here. The *Calculus of Classes* systematizes our reasoning with respect to 'classes' of arbitrary objects by defining cuts, unions etc. The complement of such a 'class' a is an absolute complement $\bar{a}$, such that $a \cup \bar{a}$ is the 'universal class'. The crucial point is that these 'classes' of the *Calculus of Classes* only contain individuals of the considered domain. There are no 'classes of classes'. The 'universal class' is just the domain considered. The 'classes' of the *Calculus of Classes* are neither sets nor classes. They obey some axioms (like Extensionality), but others (like Powerset) do not apply here. The concept SET exhibits much more complexity than the concept COLLECTION OF INDIVIDUALS!

[12] In the following paragraph "class" is scare-quoted to make clear that these collections are not proper classes, but collections of individuals.

Given the logical apparatus of **Z** we can even derive: $U = \{x \mid x = x\} = \varnothing$, *even though* we have: $(\forall x)(x = x)$![13]

There are several reasons why there is no universal set in **ZF**:

1. There is no U because this contradicts *Cantor's Theorem* (i.e. because of the Axiom of Powerset). For U we should have $\wp(U) \subset U$, but this contradicts *Cantor's Theorem* (as, trivially, a subset has at most the cardinality of the superset). [By the way: Hilbert had a similar argument working with self-mappings of functions of numbers.]
2. There is no U because this contradicts the Axiom of Foundation. For U we should have $U \in U$ against Foundation.
3. There is no U by the Axiom of Separation, as shown above.
4. As, because of further antinomies, there cannot be a set of all cardinal numbers or of all ordinal numbers – as was already clear to Cantor – there can be no U, which had to contain these sets as separable subsets.
5. There is no U by the Axiom of Pairing in combination with Foundation as $\{U\}$ could be built by Pairing (i.e. U and U again gives $\{U,U\}=\{U\}$), but $\{U\} \in U$ contradicts Foundation as $\{U\}$ does not have an element that does not share an element with it (as $U \in U$).

The absence of a universal set yields more consequences in **Z, ZF** and **ZFC**. In **Z, ZF** and **ZFC** absolute complements are missing: since subsets are separated relative to a base set the complement to a set *x* is not the collection of all things not in *x*, but only the collection of those things in the base set which are not in *x*. This follows the spirit of Separation, but violates, supposedly, our intuition as to complements. Just as Comprehension is restricted in **Z** so is complement building. There cannot be absolute complements as the absolute complement to $\varnothing$ had to be U.

As **ZF** and **ZFC** are naturally understood by the iterative hierarchy [cf. below] their definition of number cannot be Frege's. Frege used a flat universe and defined a cardinal number as the equivalence class of sets with the same equinumerosity – or a representative of that equivalence

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[13] *Proof* (Outline). If one allows for *definition by abstraction* in a pure set theory (i.e. without *atoms*, which are not sets) one has to use a scheme like the following:

$$\{x \mid \varphi(x)\} = y \equiv ((\forall x)(x \in y \equiv \varphi(x) \wedge (\exists w)(w = y)) \vee (y = \varnothing \wedge \neg(\exists w)(\forall x)(x \in w \equiv \varphi(x)))$$

Now, for an instance of this scheme with $U = \{x \mid x = x\}$, assume $U \neq \varnothing$, then the second disjunct on the right hand side is false. Therefore the first disjunct has to be true. This leads to contradiction again, by the proof for (NU). Thus the assumption has to be rejected. [In a set theory with atoms the second conjunct in the first disjunct has to be the meta-linguistic assumption that *y* is a set, cf. Suppes 1960, p.34.]

class.[14] Frege defined equinumerosity by means of bijective functions. This cannot be done in **ZF** as, for instance, there are singletons of any rank in the hierarchy, so the supposed set representing 1 had to contain elements from *any* rank, but this is impossible for a set (contradicting the Reflection Principle): Sets have a minimal rank, the rank at which all their elements are present. A collection of sets of arbitrary high rank cannot be a set, and this cannot be or represent a number.

The idea that there is no universal set seems to go against our logical intuitions as we have developed them working with quantificational logics: There is always a *domain of all objects* to be quantified over.

What then can be the semantics of **Z**? How are its quantifiers to be understood? Although there is no universal set, there is *universal quantification* in **Z**. The axioms witness this. The Axiom of Separation, for instance, says of *all sets* that for any condition the corresponding subset exists. In terms of the iterative hierarchy [cf. below] the axiom talks about sets *of any rank*.

One issue should be made clear at the very beginning: The metaphors usually employed when setting out 'the construction' of some sets, say of the transfinite ordinals, should not be taken literally as involving some *temporal* procession of *arriving* at ever larger ordinals, ranks or cardinalities. As sets are abstract entities they do not depend in their existence on any one – not even God – counting up to them. Sets are simple *there*. *All of them* are there. The metaphors of construction merely serve to express the structures the sets employ, and may serve, sometimes, as didactic devices how we come to understand some set on the basis of another collection of sets. Thus, that there is no highest rank in **ZFC** should not be misunderstood as the set theoretic hierarchy V being under construction. All sets are there, thus V *is there*. For this *ontological* thesis and corresponding universal quantification it is irrelevant whether we have epistemic means to distinguish that totality from any incredibly large, but not total collection/set.

Like **FOL**, which does not count its domain to be one of the objects to be quantified over, **Z** itself need not talk about its domain. A stronger meta-language may be used to model the semantics of **Z**, typically a second order logic (**SOL**) talking about proper classes, one of which may be the domain of **Z**. We come back to this later.[15] But suppose there to be such a

[14] In fact, in Frege's *consistent* system behind the *Grundlagen der Arithmetik* the concept BEING-IDENTICAL-TO-ONESELF should have an extension, and thus a number: the number of *all* things! The system can, however, not tell us what number this is (cf. Boolos 1987).

[15] I use "set" to talk about sets and "class" to talk about *proper classes* (so called because these classes either *are not* sets or have no *corresponding set*, both usages are common, we come back to the idea of 'correspondence' below). "Set" and "class" are

model for **Z**. What should the domain of it be called? It certainly looks like a universal set, as it comprises all sets. Then **Z** cannot be complete, since it does not deal with all collections of objects/sets. But wasn't it supposed to be complete in its application? V has to be a collection of sets, and can be no set itself in **Z**. Zermelo (cf. 1908) recognized this and concluded from the reasoning about the Russell Set that the domain of set theory 'is not itself a set'. There seem to be totalities beyond sets then.

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The standard picture of the realm of sets accompanying **ZF** and **ZFC** is, at least nowadays, the iterative or cumulative hierarchy. It can be argued that Cantor had already a conception of sets congenial to this picture, because Cantor thought of sets as build by the iterative application of set building functions. Frege's set theoretic universe, in contrast, has to be conceived as flat (non hierarchic). The hierarchy was clearly developed by von Neumann (1929), wherefore it is sometimes called "von Neumann hierarchy". Zermelo developed a similar picture in the late 1920s. The Axiom of Foundation and the Axiom of Replacement determine this picture. Foundation expresses the idea that a set occurs at some earliest level in the hierarchy (as sets are build/defined by iteration of set building operations there is some – though possibly transfinite – number of preceding set building operations). As mentioned before, talk of 'building' sets should not be taken as a process of construction, but only as an easy way to express structural dependencies between sets *all being already there*. The Axiom of Replacement expresses the continuation of ever higher levels (e.g. by collection a transfinite sequence of iterations of applying the powerset operator into a single set).

In the *pure* version of the hierarchy the starting level (or 'rank') V_0 is $\varnothing$, then there are two ways of proceeding to higher ranks

$V_{\alpha+1} = \wp(V_\alpha)$ for successor ordinals α

$V_\delta = \cup\{V_\alpha \mid \alpha < \delta\}$ for limit ordinals δ

the set theoretic universe V can then be seen as a hierarchy where later sets depend on preceding sets (although, of course, not in a temporal manner). The hierarchy is iterative as the two hierarchy building operations are applied over and over again. The hierarchy is cumulative as the sets present

thus *not* taken as synonymous here. All claims and theories referred to are adapted to this usage; formalisms/symbols are also rendered into the common format used here. Following ordinary usage equivalence sets are called "equivalence classes" although they are no classes.

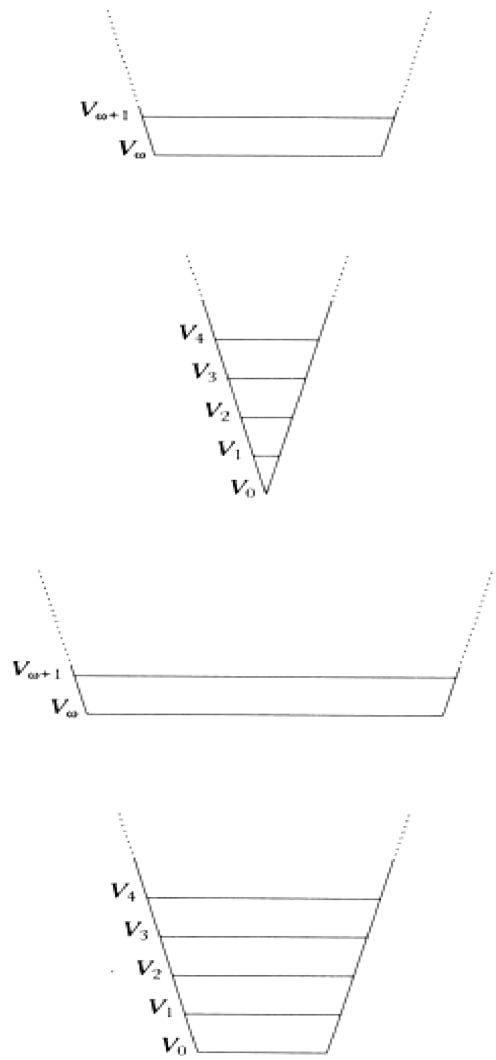
at V_α are also present at all levels V_δ with $\alpha < \delta$.[16] Each set has some earliest rank of occurrence. All ranks are transitive sets (i.e. contain all members of members of members…). The strength of the operation of collecting the powerset provides the plenty of the next stage. Reflections about how strong the idea of a powerset is concern directly the issue of the Generalized Continuum Hypothesis (GCH).

The picture is slightly different in a set theory with urelements. The set of urelements M lays at the foundation of the hierarchy $V_0 = M$. The two ways of proceeding are accompanied by the requirement that for each V_α, $M \subseteq V_\alpha$. A corresponding set theory needs to distinguish sets from non-sets and is called **ZFU** or **ZFCU**.[17]

[16] Remember that $\varnothing \subseteq V_\alpha$ for any V_α as V_α is a set. Thus at $V_{\alpha+1}$ $\varnothing$ and $\{\varnothing\}$ are present and thus each stage contains all preceding stages.

[17] Usually the system is called **ZFU**, with U being the set of urelements. The name "ZFU" may thus confuse in the context of our investigation into the existence of the universal set U. Nonetheless we stick with the usual name "ZFU" as urelements and thus **ZFU** and **ZFCU** play no vital role in this book. For us it is important to distinguish the *set* of all sets U from the *class* of all sets V, so we need the name "U" in addition to "V".

Thus one can picture V as either a pure hierarchy of **ZF**, **ZFC** (upper part in the picture) or a hierarchy based in domain of non-sets (lower part).



ZFU has a broader base than **ZF**. The dots before V_ω indicate that V_ω is the first limit level (of transfinitely many).

Z takes us with the Axiom of Inifinity to V_ω, but not to arbitrary high ranks in V. We need **ZF** (i.e. Replacement) to go further. By Replacement we know that the function in *n* for $n \in \omega$ which takes as value the n-time powerset of ω has as range *a set*, since ω is a set (by the Axiom of Infinity). Therefore (by the Axiom of Union) the union of all these powersets exists as *a set*, and thus as a next rank in V. Now we can move in **ZF** beyond $V_{\omega+\omega}$. Note also that in this rank all other **ZF**-axioms are satisfied, while –

by Foundation – the rank is not a member of itself, which establishes the independence of Replacement from the other **ZF**-axioms.[18]

Up to V_ω we find in pure set theory the *hereditarily finite sets*. They fit naturally to defining the ordinals in von Neumann's way: $n+1 \overset{\text{def}}{=} n \cup \{n\}$ and take $\varnothing$ as 0. Then in V_ω a transitive set of transitive sets is a number. We get: $n \in V_{n+1}$, $n \notin V_n$, $V_n \in V_{n+1}$, $V_n \subseteq V_{n+1}$. Ranks and numbers thus are $\in$-ordered. The hereditarily finite sets fulfil the axioms of **ZFC** save the Axiom of Infinity, although the Axiom of Choice and the Axiom of Replacement become unimportant here: The Axiom of Infinity is thus independent from the other **ZFC**-axioms. The finite system is sometimes called: **ZFC**$^{-\infty}$. In fact one could add an Axiom of Finiteness here:

$$\neg(\exists x)(\varnothing \neq x \wedge (\forall y)(y \in x \supset y \cup \{y\} \in x))$$

Obviously the Axiom of Finiteness is true up to V_ω, i.e. for all hereditarily finite sets. And equally obviously V_ω (i.e. the domain of that theory) is not finite. We meet the same situation as with Quine's basic finite arithmetic [in chapter III]. Even **ZFC**$^{-\infty}$ can do what Peano Arithmetic, **PA**, does: prove theorems concerning representability and provability (e.g. Tarski's and Gödel's theorems.[19] Note that the hereditarily finite sets provide an *intended* model for **ZFC**$^{-\infty}$ (i.e. in contrast to other unintended countable models for **ZFC**). Note also – and this may be thought to be important – that Naïve Comprehension causes *no trouble* within the hereditarily finite sets. The Russell Set, for instance, does *not exist* up to V_ω as it contains all hereditarily finite sets, since they satisfy Foundation, and thus is infinite. If the set of urelements is finite as well – as one may expect in a finite physical universe – this finite consistency of Naïve Comprehension may be the background of our intuitive support of Naïve Comprehension. Let us note this as a theorem ("y" not occurring in φ as always):

$$\text{(FNC)} \qquad |\{x|\varphi(x)\}| < \aleph_0 \supset (\exists y)(\forall x)(x \in y \equiv \varphi(x))$$

[18] Remember not to confuse the indices of ranks above V_ω with theses about the cardinality of the rank itself, the order type of its largest member or the index number occurring for the first time at that rank. $\omega+1$ (i.e. $\{2,3,4,\ldots 1\}$), $\omega+2$, $\omega+3$ etc. are, because they are order types (i.e. relational) subsets of $\omega\times\omega$, thus countable, thus sets of ordered pairs (i.e. given the usual definition of ordered pairs, sets of sets of sets of natural numbers) being subsets of $V_{\omega+2}$, members of $V_{\omega+3}$. These ranks have cardinality $\aleph_2$, $\aleph_3$ respectively and contain many, many ordinals. $\mathbb{R}$ under the usual construction (as a set of sets of sets of natural numbers) is a subset of $V_{\omega+2}$, member of $V_{\omega+3}$. $\mathbb{R}$ is uncountable, whether it has a order-type (not just a simple ordering, but a well-ordering) is not obvious and is ensured only by **ZFC**, not **ZF**.

[19] Cf. Fitting 2007. The Peano/Dedekind-Axioms for the successor function and induction follow easily in **Z** from the Axiom of Infinity. Taking natural numbers as von Neumann ordinals makes obvious that 0 is no successor and that the successor relation is functional. Induction follows since an inductive property is inherited by the successor relation, thus contains ω.

Let us leave **ZFC**$^{-\infty}$ behind and look at all ranks in V. With a little pretense we can say: In the iterative hierarchy *exists at some rank any proper subset of* V, i.e. (a) pretending for the moment that the non-set V has subsets and (b) speaking only about collections that can be sets (excluding a set of ordinals etc.). We can approximate Naïve Comprehension up to an arbitrary rank: $y = \{x \mid \varphi(x)\}$ exists for *any* φ as long as the rank of $y < \alpha$ for some ordinal α. The set y exists then somewhere below α. We can say in general: If a set x exists x has some rank.[20] Existential statements are, if true, true in parts of V. The Principle of Reflection correspondingly claims that if a general sentence or a finite collection of sentences in the language of **ZFC** is set theoretically true, there is a least rank V_α which can serve as its model (with variables in the sentences bounded to rank V_α).[21] One might expect that as all specific sets mentioned in a sentence have a rank. Limit ranks ensure this structure. Once again – as with Naïve Comprehension – we seem to *approximate* talk of *all* sets! The Reflection Principle is equivalent to the Axiom of Replacement.[22] So the fully developed picture of the iterative hierarchy established by Replacement approximates universal set theoretic talk. Unfortunately this would be too good to be true.

On the one hand we approximate universal set theoretic talk. And not just – one may claim (as Kreisel 1967 did) – set theoretic talk: Set theory can be considered to be our strongest formal system, the system to be used in the meta-theory of all other systems. Then: If some claim in some informal system is intuitively valid and can be captured in some formal system it has a set theoretical model. *Kreisel's Thesis* so states: Whatever is valid is valid

[20] *Proof* (Outline). If x existed without a minimal rank at which it exists, x would contain all ordinals as a subset, which is impossible.

[21] This does, of course, not hold for an *infinite* collection of sentences as all infinitely many instances of the schema of Replacement enforce V. The Principle of Reflection is another reason why **ZFC** cannot be finitely axiomatized: If **ZFC** could be finitely axiomatized, then it would establish – by the Principle of Reflection – a model of itself, thereby establishing its own consistency, contradicting *Gödel's Second Incompleteness Theorem*.

[22] *Proof* (Outline). The Reflection Principle entails Replacement, since if the antecedent of Replacement is true, there has to be a rank V_α modelling it; the set postulated as existing in the consequent of Replacement will be a subset of that modelling rank V_α. Replacement entails each instance of the Reflection Principle in going through the quantifiers of the finitely long compound $(\forall x)\varphi(x)$ taking the lowest possible rank of satisfying instances (which have to be there to make $\varphi(x)$ true) and uniting them and their dependencies (by a Replacement function) into a highest most comprehensive rank, which thus models $(\forall x)\varphi(x)$. ■ Omitting the Replacement schema and restricting separation to formula φ with quantifiers bounded to some set provides a further weakened theory $\mathbf{Z}^-$ (also known as 'MacLane Set Theory'), which nonetheless proves sufficient for most of mathematics.

in a set theoretical model, and if – as we may suppose – finitely many sentences were used in that piece of reasoning, it is valid at some rank V_α.

On the other hand, however, we, obviously, *shift the domain of reference* from V to some rank V_α. So a universal statement (say, the Axiom of Pairing) does no longer talk of all sets, but only of those up to V_α. Seen in this light the Principle of Reflection resembles the *Löwenheim/Skolem-Theorem* in allowing for non-standard or *unintended* models of universally quantified set theoretic sentences. As V_α can be arbitrary high one may see this as less concerning than the countable models ensured by the *Löwenheim/Skolem-Theorem*. If V_α is a sufficiently high transfinite rank we approximate universal talk. We can also understand the possible shift of domain of reference as underlining the insight that universal set theoretic talk is bound to *strong* axioms like Replacement.

The universe V is not reached by any ladder ('construction principle') used within it. It is as strongly inaccessible by such steps as it can be. Otherwise we only have a temporary halting point V_α. V is no number, is no set, no union or power of sets. V can only be thought as *sui generis*. How do we know this? Because otherwise it could be superseded in one of the usual ways. We thus have a *transcendental argument* concerning V's nature: it cannot be otherwise, since otherwise it wouldn't be.

Without the Axiom of Foundation or endorsing an Anti-Foundation Axiom the realm of sets is larger containing with the unfounded sets more collection like entities. Where are *these* collections collected in? U seems a good candidate for an collection of unfounded collections as U∈U itself. But unfortunately **Z** forbids U. Are unfounded collections sets? Or does our concept SET entail that sets are grounded collections? In this case we had the problem that on the one hand we had to endorse the Axiom of Foundation, but this excludes U from our set theory. If sets are abstract entities nothing seems to exclude that they contain themselves as all spatial images are inappropriate. Picturing non-wellfounded sets by graphs (cf. Aczel 1988) shows easily membership bending back to its origin. Anti-foundationalist set theories contradict our concept of set, however, if set identity becomes more than identity of membership (cf. Aczel 1988, chap.4). The iterative hierarchy motivates our picture of sets as well-founded by stressing the idea of ontological structural dependence between a set and its members. In this light a set containing only itself, $x = \{x\}$, seems unnatural. U, in contrast, contains besides all other things itself. We might recognize U as a set *sui generis* and allow for U what we do not allow for other sets. Foundation would make an exception for U. But the exceptions would not end here as U, being subject to the others axioms if still a set, is exceptional – even inconsistent – with respect to *Cantor's Theorem*, for instance. Foundation certainly is built in the iterative hierarchy and V does not pose the problems with respect to Foundation that

U does. According to the story of the iterative hierarchy, unfounded sets do not exist. The Axiom of Foundation follows from the set up of the cumulative hierarchy. The two conditions to proceed to higher ranks ensure the axioms of Pairing, Sums, Powerset and Infinity. Coupled with the idea of sets being extensional the structural properties of the iterative hierarchy thus *entail* the **ZF** axioms (cf. also Boolos 1989).

There are – besides the question of an Anti-Foundation Axiom – incompatible set theoretic axioms (like the Axiom of Choice *vs*. the Axiom of Determinacy[23]), which shows that there are related realms of set-like entities (sharing the basic axioms), but which cannot be consistently united. There might be a unified inconsistent realm of all these sets [cf. Chap. V]. Even the incompatibility need not show that *our* concept of set is not settled. One of the set theories may be thought to be more natural. Even a concept SET settled in its basic aspects (like set separation and powerset existence) may leave some questions unsettled. The (Generalized) Continuum Hypothesis is the best known example. The simple Continuum Hypothesis [$\neg(\exists x)(\aleph_0 < |x| < 2^{\aleph_0})$] is even independent of the Axiom of Choice.[24]

V has sets of arbitrary high rank. V itself does not occur in the hierarchy itself. V taken as the proper class of all ranks in V is a model of **ZF**. If V exists **ZF** is consistent, as V satisfies all its axioms. Large cardinals (strongly inaccessible cardinals beyond the reach of any set building

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23 Cf. Jech 2003, pp.627-43. The Axiom of Determinacy in so-called 'Descriptive Set Theory' contradicts the Axiom of Choice, what one may take to be bad enough. It also entails some strange results for large cardinalities (like $\aleph_1$, $\aleph_2$ being measurable cardinals, but $\aleph_3$ … not being measurable). **ZFC** seems closer to our conception of sets in this regard.

24 *Proof* (Outline). Alephs are defined as infinite well-orderable cardinals. The Axiom of Choice is equivalent to the statement that any infinite cardinal is an aleph (as it implies the Well-Order Principle). Negating the Axiom of Choice (and thus the Well-Order Principle) one may endorse the simple Continuum Hypothesis *but* maintain $2^{\aleph_0} \neq \aleph_1$, since one may now deny that the Continuum can be well-ordered, whereas the *combination* with the Axiom of Choice entails $2^{\aleph_0} = \aleph_1$, since the Axiom of Choice entails that any infinite initial ordinal is an aleph. ■
Cantor proved in 1883 that there is no cardinality between the cardinality of the collection of finite ordinals ($\aleph_0$) and the cardinality of the collection of all countable *well-orderings* of ω, that cardinality thus being the next well-orderable cardinality: $\aleph_1$. Given the Continuum Hypothesis $2^{\aleph_0}$ is the cardinality of all countable *linear* orderings of ω. Given the Well-Order Principle thus $\aleph_1=2^{\aleph_0}$.
The Generalized Continuum Hypothesis (GCH) entails the Axiom of Choice: Using the first aleph GCH claims for all infinite cardinals $x = 2^{y+\aleph_0}$. $x = 2^{y+\aleph_0}$ entails *x* being an aleph, which makes *y* an aleph. The GCH thus excludes that there are cardinals in between well-orderable cardinals (i.e. in between well-orderable sets), so that *via* its respective cardinal number any set can be well-ordered. ■ (On arguments and intuitions around CH and GCH cf. Potter 2004, pp.266-82; Maddy 1988, §2.)

operation by being uncountable, regular and greater than 2^δ for any preceding cardinal δ), if existing, are such models as well. For V the axioms of **ZF** are construction principles and thus trivially satisfied. For (strongly) inaccessible cardinals the important observation is that they are assumed to be just larger transitive sets. Take the least such cardinal; any function within it is of lower rank as the cardinal itself; thus the range of the function is a set, which has this least inaccessible cardinal satisfy the Axiom of Replacement – the other axioms are obviously satisfied again (cf. Jech 2003, pp.165-67).

Having *all* subsets of a rank present at the next rank suits the Axiom of Choice: If a family of non empty sets *x* exists at some rank V_α, the members *y* of that family exist already at lower ranks V_δ with $\delta<\alpha$, and their members *z* exist already at lower ranks V_γ with $\gamma<\delta$ (relative to a δ for some *y*); thus as these *z* are elements of some V_γ a *set w* containing one of them for each $y \in x$ exists (at the latest) at the rank V_α of *x*. Choice is natural in the iterative hierarchy. V rather corresponds to **ZFC**.

Once we have one of the inaccessible cardinals or the class V of all sets we have a model of ZF and could be content with respect to our theory of sets. So should we care about their nature?

Leaving V to the side for a moment let us consider large cardinals. We have just talked about them, so we know something about their nature and we can ascribe properties to them. So they should be the objects of some theory.

Zermelo thought of strongly inaccessible cardinals (his 'Grenzzahlen') forming themselves an unbounded sequence. This, however, implies that we quantify over them, and are again in the situation of asking over what domain now our quantifiers run. Is this collection of *Grenzzahlen* itself some *Grenzzahl*? Supposedly not to avoid antinomies of the Burali-Forti-type. Then again if we now introduce *Super-Grenzzahlen* we can start all over again with them – and once more the whole process iterates. Zermelo thought: 'This series reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, …' (1930, p. 47).

Now, this way of thinking may be innocent for a constructivist, but for a set theoretic realist the idea that sets have to come into existence is simply wrong. Placing them at some rank in the hierarchy does not mean that they come later (in time?) than the other sets. Frege's universe is anti-foundational. And for a Platonist an anti-foundationalist universe has the advantage of keeping all ideas of stepwise construction at bay. As all abstract objects are there they exhibit some ontological dependencies, but this does not require that some are before or beneath others. Impredicativity is no problem in such an anti-foundational universe. Zermelo himself rejects any spatiotemporal associations. A well-ordering 'has nothing at all to do with spatiotemporal arrangement'. He also thought the term "choice" to be problematic as one may associate (temporally) successive choices being performed, where we have only a representational/selectional correlation (cf. Ebbinghaus 2007, p.69, 135). The ranks express a structural dependency only. All ranks are there. In the same way all of that coming beyond the set theoretic ranks (i.e. any large cardinal) *is there*. Thus there should be *a collection of it all*. Assuming a sequence of large cardinals thus does not seem to solve the problem of collecting sets, but either adds the issue of an *incomplete universe* [cf. Chap. VI] or means that V contains them all and only our *set* theory, say **ZFC**, is *not complete yet* and has to be strengthened by further axioms.

The issue of large cardinals is independent from that of the universe of sets. If one can argue that some idea of some type or large cardinals comes from our concept of sets – say, why should $\aleph_0$ be the last inaccessible? – then these large cardinals may be thought of as stages in V above those which **ZFC** (so far) treats of. *Any* type of closure operation on preceding collections should correspond to *a set* within V. This idea resembles the content of the Reflection Principle: Any finitely specified closure condition can be modelled by some rank. Large cardinals may provide a universe and a model for **ZFC**, but they differ from classes in being collectable themselves and thus being members of the overall universe of (extended) set theory. Another argument for such additional sets stems from Scott's proof that V≠L given large cardinals, as the notions of (unrestricted) powerset and uncountability stand in conflict to V=L. The constructible universe seems unnatural, even though V=L entails the Axiom of Choice and the Generalized Continuum Hypothesis, excluding it speaks in favour of large cardinals. The constructible universe violates the idea of purely *extensional* sets inasmuch as pure extensionality should allow for sets beyond any descriptive powers. One might think that it follows the idea of Naïve Comprehension, that sets correspond to properties, but why should all objective properties correspond to formulas in the first place? Proceeding to the next rank by the full powerset operation suits the simple idea of the powerset. Curtailing the powerset to subsets which are definable leaves out sets that should *be there*.

V is the ultimate model of the universe also in the sense that constructions like 'forcing' or means of building 'inner models' start from V (cf. Arrigoni 2007; on the formal details cf. Jech 2003, pp.175-223).[25]

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[25] Leaving here to the side the problem that such models are non-standard or unintended, e.g. in being countable; cf. the remarks in the next chapter on limits of expressivity. One may add that inner models like L, which restrict the powerset operation, but satisfy the others axioms in their standard reading (relative to the shrunken universe), are less non-standard than models generated by forcing.

The iterative hierarchy does not know several classes. It might be preferable not to call V a class, but to treat of V as a very special object in its own right – an issue of axiomatic ontology. If we call V a class it is not to be thought of in the manner of **NBG** or **MK**, since there is no part of set theory which addresses it, like Comprehension and Limitation of Size address classes in **NBG** or **MK** [cf. chap. II below]. V is not in the range of set theoretical quantifiers. It is not in the domain. Calling V 'a class' in the context of the iterative hierarchy and **ZFC** means there exists only one class (outside of our theory of sets).[26]

V *is* the range of the quantifiers in **ZFC**. Cantor claimed that every potential infinite presupposes an actual infinite 'and cannot be thought without it' (cf. Cantor 1887). This is the Domain Principle: Speaking of and quantifying the *x* presupposes the domain of the *x*.[27]

V is a very special entity, both within the picture of the iterative hierarchy as in our meta-theory modelling our theory of sets. V has no subsets as V is no set. V is not well-ordered – even in the presence of the Well-Order Principle only *sets* are well-ordered. V is not the domain of a (replacement) function, sets are – and so on. V contains all ordinals and all cardinals, but there is neither a set of all ordinals nor a set of all cardinals. They cannot be established as subsets of V, since V is no set (and thus Separation does not apply to it).

For V to be more than a stopping point to be superseded V has to be an entity *sui generis*. This means informally that V is exactly what the picture of the iterative hierarchy shows it to be. V is *determined*, not indefinite, and *unique*. Formally this means

- that V cannot be an element of whatsoever other collection – on pains of re-introducing distinctions of the set/class-type
- that there are no other entities of V's type (not a collection of proper classes)
- that V is an entity which can be talked about by its name, without including it into a domain of reference.

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[26] At some time Cantor considered distinguishing several 'absolutely' large, 'inconsistent' collections (like those of all ordinals or all cardinals). But they play no role in a transfinite set theory based on standard logic. Even apart from producing antinomies, these collections play no indispensable role in proofs about sets. So Cantor came to consider *the single* absolute, inconsistent totality beyond any further increase.

[27] Cf. Moore 1990, pp. 114-22; Tiles 1989, pp. 95-107. The principle sometimes – ignoring Cantor? – is discussed as 'All-in-one Principle', going back to (Cartwright 1994).

V is not a standard object of (set theoretic) model theory. The only thing V 'does' is containing all the sets. A universally quantified sentence of pure set theory is meaningful as there is an entity which provides all the variable values: V.

A unified language has to distinguish urelements, sets and V. Again: V cannot be unified with them in a domain. The name "V" refers to V rigidly. End of story.

II

LIMITATIONS OF SIZE

The idea of classes already mentioned we may take the universe of **Z** (and other related set theories like **ZF**, **ZFC**) to be a class. We have to have formal resources then to distinguish classes from sets. Let us use "V" to denote the *class* of all sets (by what ever condition it may be identified, i.e. whether by "x = x", for *sets x*, or something else). V is universal for sets.

As "U" was used above to denote a universal set, V ≠ U as long as classes and sets are kept apart.

Von Neumann (1925) introduced a new axiomatization of set theory – originally working with functions instead of collections – distinguishing proper classes from sets.[28] Von Neumann explicitly states the axiomatic approach now obligatory in set theory:

> To replace this [naïve notion of set] the axiomatic method is employed; that is, one formulates a number of postulates in which, to be sure, the word "set" occurs, but without any meaning. Here (in the spirit of the axiomatic method) one understands by "set" nothing but an object of which one wants to know no more than what follows about it from the postulates.

In **NBG** classes are distinguished from sets by the Limitation of Size Axiom. It says:

(LSA) A class is not a set if and only if

there is a bijection between it and the universal class V.

The 'universal class V' is, again, *universal* for sets only. Thus there is only one size for classes. Some collections are too big to count as sets, therefore the name of the axiom. All other improper classes may be either taken as

[28] The present form of the theory resulted from further development and employment by Bernays (1937, cf. 1968) and Gödel (1940) and therefore is called "NBG".

being sets themselves or having a *corresponding* or *representing* set (cf. Bernays 1968, p.63), which has the same elements as the improper class.[29]

Limitation of Size entails the Axiom of Choice: because there cannot be a set of ordinals, the collection of ordinals has to be as large as V, which, therefore, can be well-ordered. Limitation of Size in itself does not exclude a multitude of proper classes. It excludes a multitude of cardinalities beyond |V|.

Limitation of Size by assuming V to be a class works with the idea 'one size fits all (classes)'. All classes have *the same size*. Even if we grasp the idea that some size is too large to be collected into a set, why should we stop the idea of oversized collections to continue? Why shouldn't there be some operations or some structure with respect to classes that provide super-classes? And if there is some intuitive/conceptual appeal to the idea of Limitation of Size, why not stop earlier? Why assume $V_{\omega+\omega}$ or even V_ω as *sets* in the first place?

Limitations of size are unspecific or open as to which size might be taken as limit. Randall Holmes 'Pocket Set Theory' (**PST**) uses $\aleph_1$ as limit (i.e. the universe has cardinality $\aleph_1$ and all infinite sets within that universe have cardinality $\aleph_0$). So **PST** has the Continuum Hypothesis built in. **PST** (cf. Holmes 2006, §9.1) has the axioms: Extensionality, (SC*) of **MK**, Existence of ∅ and Singleton {x} for a set *x*, (unordered) Pairs, Existence of Relations (i.e. ordered pairs); the Axiom of Proper Classes, that all proper classes have the same size, is the **PST** version of Limitation of Size; the Axiom of Infinity not just degrees an infinite set, but demands all infinite sets to be of the same size, $\aleph_0$. The Powerset Axiom is, of course, missing: the powerset of an infinite set is a proper *class* (of cardinality $\aleph_1$), no further power can be generated. The Russell Class is a proper class in **PST** as well. And as the ordinals are a proper class the universe of **PST** can be well-ordered (i.e. Choice and Replacement follow *as theorems* from the mentioned axioms, just as in **NBG** and **MK**).

What **PST** lets us see by all this is that any stopping point in limitations of size is arbitrary. **PST** suffices for a lot of mathematics, and even moving upwards a few alephs still has us positing some arbitrary limit, as long as the limit is assumed to be of the kind we know from **NBG**, establishing an uncollectible collection of equinumerous collections of a new kind.

Limitation of Size, thus, seems unnatural. Let us come back to the issue of Comprehension. The formal language of **NBG** and **MK** uses typically (at least) two sorts of variables and constants: one type for sets, the other type for classes (and may be other types for atoms…). **NBG** might be seen as

[29] As I use "class" for proper classes in distinction to sets, I use "improper class" here for collections *X* that are bound by class variables in **NBG**, but have a 'representing' set *x* such that $(\forall y)(y \in x \equiv y \in X)$.

further evidence for the thesis that the problem behind the antinomies rests not in Comprehension but in universality. Bernays (1937) and Gödel (1940) derive within (their syntactic versions of) **NBG** a Comprehension Principle (as a schema in φ) for sets:

(SC) If φ does not contain bound class variables,

$$(\exists Y)(\forall x)(x \in Y \equiv \varphi(x))$$

This principle of Set Comprehension (SC) collects sets (lower case variables) in a proper or improper class (upper case variable) corresponding to a defining property φ. "x=x" gives us V, "x≠" ∅ etc. The improper classes will have a representing set then. Further on, **NBG** is a conservative extension of **ZF**; in the language of **ZF** the two systems have the same theorems. As **NBG** is a stronger theory than **ZFC** one can use it to describe a model of **ZFC**. In a sense we understand what **ZFC** says and see the consistency of **ZFC** in such a model (given, of course, that **NBG** itself is consistent). As V provides a model of **ZFC,** showing it to be consistent, an extension to $V_{\kappa+1}$ taking V, say, as the first strongly inaccessible cardinal V_κ and $V_{\kappa+1} = \wp(V_\kappa)$ shows **NBG** to be consistent! The difference $\wp(V_\kappa)\backslash V_\kappa$ then contains all proper classes. And there are then many, many more proper classes than sets. If classes are understood as collected by formulas, there cannot be more classes than formulas, there being thus less classes than sets. One then has to forsake collecting subclasses, which should exist, since there elements exist, into a collection – etc. As classes are supposedly larger than sets the whole conception of tying classes to formulas seems to add new peculiarities to the old ones.

NBG thus contains Comprehension as well as universality. Of course **NBG** contains only universality for sets. Universality comes at the prize of accepting a new ontological category: (proper) classes. Classes themselves are understood inasmuch as one can give the axioms of a theory as class axioms and then provide corresponding principles for sets. The main drawback, however, is the well known shifting or elevation of the problem of universality to classes. We have a class of all sets, but, of course (i.e. on pains on versions of the set theoretic antinomies), we have no class of all classes. (Often we have *not a single* class of classes as classes are taken only as containers and never as elements.) This is as disturbing as the original problem with U, one might think. Certainly classes should obey: $(\forall X)(X = X)$. Therefore there should be a collection of them – what are the class quantifications running over, anyway?

> If quantification is throughout being treated extensionally, then to admit quantification over classes is to presume that the classes of sets form a determinate totality which ought itself to be admitted as a class, a class to which all classes of sets belong. (Tiles 1989, p.130)

*

Morse/Kelly-Set Theory (**MK**) differs from **NBG** in having a unrestricted Comprehension Schema (with a class variable X)

(SC*) $(\exists X)(\forall y)(y{\in}X \equiv \varphi(y)$

where in φ one may have bound class variables as well as set variables. The schema is thus – in distinction to **NBG** – impredicative. **MK** and **NBG** share Limitation of Size. **NBG** does not extend **ZFC** in the language of sets, **MK** does. The extensions of **ZFC** brought about by **MK** can, however, also be proven in **ZFC+**(some large cardinals). Proper classes are not needed to deliver new theorems about sets.

In fact, as there are finitely specific operations to built sets from sets (like complements, products, cuts…) **NBG** can be finitely axiomatized (by substituting more specific set building axioms for the schematic (SC), which can then shown to be valid) whereas neither **MK** nor **ZFC** can be finitely axiomatized, because of the schematic character of (SC*) and Separation and Replacement (cf. Cohen 1966, pp.73-78; Montague 1961).[30]

Both **NBG** and **MK** endorse the important existence claims:

([∈]) $(\exists X)(X=\{\langle x,y\rangle \mid x{\in}y\}$

([=]) $(\exists X)(X=\{\langle x,y\rangle \mid x = y\}$

There is a class representing the membership relation (between sets). This suits the idea of having membership (represented by "∈") as the basic relation in set theory. The absence of these collections seems as unnatural as the absence of U. The collection [=] can stand in for U – as U does not exist in **NBG** or **MK**, since a set cannot be that large.

Both theories are two-sorted first order theories having, for instance, non-intended countable models.

In their intended interpretation some see in them a solution to our quest for U – and/or [∈]. The major problem with this view is their unaccounted use of the multitude of classes, which are quantified. The realm under consideration needs to include both representable classes (and their corresponding sets) as well as the proper classes. What is it? V – as understood by **NBG** and **MK** – is just a member of *this* domain!

One may even have classes within classes, as long as Comprehension (SC) only applies to sets. In Ackermann set theory (cf. Ackermann 1956) classes, therefore, cannot be distinguished from sets by being non-members. Each class, for instance, has its singleton.

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[30] Of course 'finitely axiomatized' always means 'consistently axiomatized': any system using **FOL** can be inconsistently finitely axiomatized by the axiom: ⊥.

The notion of set cannot be *defined* in this theory. Ackermann uses a comprehension principle restricted to sets:

(ASC) $(\forall x)(\varphi(x) \supset Set(x)) \supset (\exists y)(\forall x)(x \in y \equiv \varphi(x))$

As "Set(x)" is a formula we have:

(1) $(\forall x)(Set(x) \supset Set(x)) \supset (\exists y)(\forall x)(x \in y \equiv Set(x))$

So we have some collection (i.e. a proper class here)[31] containing all sets, although the notion of set cannot be defined! Ackermann's set theory with respect to the well-founded sets turns out to be equivalent to **ZF**. It differs from **NBG** in having not all of **NBG**'s proper classes (e.g. no proper class of ordinals, cf. Holmes 2006, §5; Lake 1973).

If one allows for proper classes, but then proceeds (indefinitely) beyond classes, as classes should be collectible themselves (e.g. Blau 2004), one should never have made a set/class-distinction in the first place. A stopping point in the progression of collections can only be a unique entity of a different kind.

We come back to the class/set-distinction in chapter VI. Whatever virtues and beauties **NBG** and **MK** provide they do not solve our problem of universality.

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The picture and the criticism do not change much, when we turn to Second Order Logic (**SOL**). Zermelo himself always aimed at a categorical characterization of sets. In both his systems he used **SOL**, claiming this to be more natural than a first order version of **ZFC** and in awareness of the downward *Löwenheim/Skolem-Theorem*. **FOL** is sound and (strongly) deduction complete, but **SOL** allows for categorical models.[32] As all properties of infinite sets are structural this (identity up to isomorphism) is perfectly fine. The second order version of **ZFC** is **ZFC2**. Since **ZFC2** is categorical an **ZFC** open question like the Continuum Hypothesis is settled by the **ZFC2** models – if it has any, of course.

[31] In Ackermann Set Theory there are non-sets. *Proof* (Outline). Argue indirectly starting with one of the antinomies, say the condition "$x \notin x$" and assume "Set(y)" for the collection *y* of sets $x \notin x$, arrive at: $\neg$Set(y).

[32] Note that the soundness of **FOL** does not automatically carry over to first order **ZFC**, as **FOL** is proven sound with respect to set theoretical models/domains. For parts of V there are – by the Reflection Principle – models for the restricted axioms (namely some higher ranks). V itself cannot be taken as a domain in the fashion of **FOL** soundness proofs.

ZFC2 can express the concepts of finitude and infinity:

(INF) $(\exists f)((\forall x)(\forall y)(f(x) = f(y) \supset x=y) \wedge (\forall x)(X(x) \supset X(f(x))) \wedge (\exists y)(X(y) \wedge (\forall x)(X(x) \supset f(x) \neq y))$

is open in "X" (resembling the Axiom of Infinity with "*f*" denoting the successor function). The negation of this sentence expresses finitude (FIN).

Models of these two sentences (INF) and (FIN) have to be infinite, respectively finite; there are no non-standard models or versions of the *Löwenheim/Skolem-Theorem.*

The Continuum Hypothesis then becomes:

(CH_2) $(\forall X)(INF(X) \wedge X \subseteq \wp(\aleph_0) \supset |X| = \aleph_0 \vee |X| = \wp(\aleph_0))$

This is either *true* or *false* in the **ZFC2** models. It does not follow from the other axioms though.

ZFC2 is the system consisting of:

- **SOL** (extending **FOL** by rules for introducing and eliminating second order quantifiers); identity is not a logical constant "=", but defined, $x = y \overset{\text{def}}{=} (\forall X)(X(x) \equiv X(y))$.
- The **ZFC** axioms: Extensionality, Foundation, Pairing, Sums, Powerset and Infinity.
- The Axiom of Replacement: $(\forall f)(\forall x)(\exists y)(\forall z)(z \in y \equiv (\exists w)(w \in x \wedge z = f(w)))$
- The Axiom (Schema) of Comprehension: $(\exists X)(\forall y)(X(y) \equiv \varphi(y))$ [where "X" is not free in φ]
- The Axiom of Choice: $(\forall X)((\forall y)(\exists z)X(y,z) \supset (\exists f)(X(y,f(y)))$

Replacement says that the restriction of *any* function to a set gives *a set* as range. Replacement – as always – allows deriving Separation, the Axiom of Comprehension is, of course, much stronger. Replacement is no longer a schema, Comprehension is. This system **ZFC2** is equivalent to **MK** [cf. (SC*) above]! It has to be strengthened piecemeal to attain a more inclusive character (e.g. by adopting CH_2) although it will never be deduction complete.

Full blooded **SOL** is more than two-sorted **FOL** (with one sort of variables for individuals/sets and one for collection of them). In a pure set theory **ZFC2** takes sets as individuals and *all* collections of them as values of the second order variables (cf. Shapiro 1991). Otherwise, as with **FOL**, the *Löwenheim/Skolem*-Theorem applies and there are countable models. **ZFC2** if not just a many sorted first order language quantifies over the properties used in first order **ZFC**. **ZFC2** is thus a property theory. Comprehension says there is a property corresponding to any (first order) open formula in the language, only by an extensional perspective can these quantifiers be read as referring to classes. The second order variables thus are in this extensional perspective not ambiguous: "X(y)" can be read as *y* having either the property or belonging to the class *X*. One could also distinguish within the first order variables between urelements and sets. Some of the collections of sets are sets themselves. As in **NBG** one might single our those sets collections which correspond to a (representing) set (like in **NBG** classes can – but need not – correspond to representing sets). The domain of the first order variables in **ZFC2** has to be the collection *of all sets*, the second order variables ranging over subsets of *this domain*. If one models **ZFC2** with a typical meta-theory this meta-theory may be a set theory of the kind **ZFC+** (e.g. with some axioms postulating large cardinals). In case the domain of individuals is taken as the first (strongly) inaccessible cardinal δ the second order variables range over the *powerset* of δ, as even inaccessible cardinals are introduced as sets, although larger ones.

There is another interpretation of **ZFC2** though. One may take V as the range of first order variables (or at least those of the set type). One may take subsets of V as the range of second order variables. As V must not be taken as a set, there is no need to submit it to the Power*set* Axiom. Of course the meta-theory in this case will not be modelled in a set theory of any strength, but if V is an entity *sui generis* one may expect so. Set theory is the strongest formal system, since we use it or could use it to model other formal systems. Therefore we may allow it to be special in describing its way of reference and variable binding [cf. Chap. VI for further discussion].

Plural quantification (as popularized by Boolos 1984, 1985) avoids introducing a collection of classes. It thus fares better in adding no new riddles than **MK**-style theories. Plural quantification, however, risk falling back to the strength of a two-sorted **FOL**. More importantly, the issue of V cannot be dissolved this way. Even if the uppercase variables (formerly known as 'class variables') bind individuals together as group without at the same time building a new collection – quite a feat, one might think – the individuals still come from a presupposed domain: The plural quantifiers share the feature of first order quantifiers of picking out objects *out of* that domain. The domain issue does not go away by just presupposing the domain or reverting to a stronger meta-theory again.

Turning to **SOL** so does not solve in itself our problem of universality and an unaccounted set/class-distinction. **SOL** and **ZFC2** (e.g. as used by Shapiro 1991) are even worse than **NBG** or **MK**, because identity is defined *only* for individuals (i.e. sets), as is clear from the definition: $x = y \stackrel{def}{=} (\forall X)(X(x) \equiv X(y))$. So classes cannot be identified. The background theory assumes an extensional understanding of the second order variables (equating predication, X(y), with membership, y∈X), but extensionality is not expressed within the language. Classes are thus a sort of collections different from sets not only in size (as in **NBG** and **MK**) but also in character. One can add extensionality by the axiom:

(EXT$_2$) $(\forall X,Y)(X=Y \equiv (\forall x)(X(x) \equiv Y(x)))$

Before we, thus, leave **ZFC2** behind we have to consider shortly two arguments that we have *no other choice* in set theory than turning to a higher order logic.

It is sometimes said that to accept a schema presupposes endorsing its universal closure (supposedly in a stronger meta-theory). Even if that was true – which is not obvious as Hilbert's use of indefinite expressions may show – it would not commit us to **SOL** in case of **ZFC**. The quantification can concern formulas in the language of **ZFC**. Transforming **ZFC** into a many-sorted system looks like substituting a second order system (by employing a second group of quantifiers) for **ZFC**, but without full blown second order semantics we stay with **ZFC**; as there are countable many formulas there is – in light of the 'schema'-argument – no need to revert to full blown second order semantics. We reach a sort of Henkin-semantics for a many-sorted version of **ZFC** (i.e. with respect to the second order variables).

The argument in favour of higher order and second order logic is, further on, often put in terms of expressive power.

Many concepts are said to be inexpressible in **FOL**, for instance: FINITUDE, WELL-ORDER, UNCOUNTABILITY. There are formula which seem to express these concepts, for instance

$|x| < \aleph_0$ and $|x| \geq \aleph_1$

express that the cardinality of x is less than $\aleph_0$ (i.e. x is finite), respectively at least $\aleph_1$ (i.e. x is uncountable). Both are available in first order **ZFC**. The claim that they do not express what they seem to express appeals to the *Löwenheim/Skolem-Theorem* that any first order theory has models which are countable and models or arbitrary infinite size. As the two expressions – and in fact any theorem and axiom of **ZFC** – can be made true in such models, they do not enforce that the structure corresponding to **ZFC** has the properties the formulas of **ZFC** talk about. In this sense concepts like FINITUDE and UNCOUNTABILITY are inexpressible in **ZFC**. As **SOL** is categorical in its models, any property it expresses some structure as possessing is enforced on the models. See the two sentences (INF) and (FIN) above. **SOL** in this sense is able to express, for instance, FINITUDE and UNCOUNTABILITY. The *Löwenheim/Skolem-Theorem* does not hold for **SOL**. Given that sense of expressivity the only adequate set theory may be taken to be **ZFC2**. *Unless* – still following this line of reasoning – one incorporates very large cardinals (like ‘measurable cardinals’) as these aren’t even uniquely 3rd order describable (cf. Jech 2003, p.295)!

Now, the models which spoil the work of formulas like the two above are clearly *unintended models*. As a reply to the argument in favour of higher order logic this is not as weak as an appeal to an easy *recognition* of intended models seems. It is not just so that – as logicians like Russell and Frege or Lesniewski in *pre*-model theoretic semantics time sometimes said – the formulas of our formal language carry their (intended) sense on their sleeves; a view which only the now omnipresent distinction between *pure* syntax and (almost arbitrary) interpretation casts doubt on. In case of set theory – as in case of many other logical systems – we have a *clear picture* of the intended interpretation: in case of **ZF** – the iterative hierarchy [the picture just outlined in Chap. I]. The iterative hierarchy is a model for **ZF**, it not only makes its axioms and theorems true, but it makes them true in their intended interpretation – *including* those formulas which claim a set to be finite, uncountable, well-ordered etc. Given the picture provided by the iterative hierarchy the case against first order set theory based on expressive limitations seems rather weak.

On the opposite, one may raise worries about **SOL**. In a wide sense of “logic” many formal systems are logics, sometimes just a set of formulas with closure conditions is taken as ‘a logic’. Logic as related to (human) reasoning requires a much narrower sense of “logic”. A logic, roughly, models some ways in which (human) reasoners derive consequences from premises. The modelling typically employs a formal system with an explicit syntax and semantics. A constraint on logics in this sense follows from the fact that humans are finite reasoners – at least in their earthly life. Therefore they can draw inferences only from a finite set of premises. If a consequence follows from a set of premises, it has to follow from a finite subset of these premises: compactness. A logic in the traditional, not technically liberalized, sense has to be compact. As **SOL** is not compact, it is no logic in that sense. So whatever else may have been arguments in favour of **SOL** they are to be set aside.[33]

Within standard logic and an object-/meta-language distinction there might be a fruitful division of labour: the *logic* used will be a first order system, and the meta-language will contain a *categorical* description of the *intended model* (as any way of spelling out the intended model, even talk of

[33] At one point Shapiro (1991, pp.50-53) seems to see the problem, but proposes only vaguely to keep **SOL** semantic validity and justification in correct inferences apart, which at least sounds like forsaking capturing logical reasoning in a formal system. On some second thoughts by Shapiro cf. Shapiro 2003.
It may also be worth reminding here that the non-standard models of **PA**, used as toy examples of a side-effect of compactness in introductory logic classes, have the unnatural features that “+” and “*” cannot be generally recursive in them.

intentions concerning reference to the iterative hierarchy, in a first order language can be re-interpreted again).

Whatever virtues **SOL** as a linguistic framework has, whatever beauty second order descriptions may have from (a) God's point of view, the human concept SET cannot be bound in its *logic* to them.

By way of comprehension that logic naturally provides U, so that ideally we incorporate U at this level (i.e. not at some meta-level somewhere else).

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Let us, finally, look at another theory beyond the size limitations of set theory: Category Theory.

Category Theory wants to be even more abstract than set theory. Sets are just one category among many. According to its 'official history' category theory was invented as a branch of foundational studies because of the size limitations of standard set theory **ZFC**. One wanted a theory dealing in collections which unify all – really all, one might say – of a kind, especially *all sets*, even *all categories*.

Many introductory books in category theory – like many in set theory – follow a more or the less naïve approach. They introduce a category of sets (cf. Lawvere/Schanuel 1991,pp.13-21). This includes a domain and an identity map. This domain has to be U and the identity map then has to be a function f:U→U. Thus one seems to have U with its problems (e.g. a function f:U×U→{0,1} representing membership).

Self-reflective category theory distinguishes categories from 'meta-categories'. Meta-categories are in some sense 'larger'. Mac Lane (1998, pp.7-26) introduces a category of sets with a 'universal set' U_C. This is defined as the closure of several set building operations:

(i) $x \in u \in U_C \Rightarrow x \in U_C$

(ii) $x \in U_C,\ y \in U_C \Rightarrow \{x,y\} \in U_C,\ \langle x,y \rangle \in U_C,\ x \times y \in U_C$

(iii) $x \in U_C \Rightarrow \wp(x) \in U_C,\ \cup x \in U_C$

(iv) $\omega \in U_C$

(v) If f is surjective $f{:}a \to b,\ a \in U_C,\ b \subset U_C \Rightarrow b \in U_C$

U_C conspicuously is not defined as {x | x = x} and it is explicitly excluded that $U_C \in U_C$.

On closer inspection one sees that the crucial condition (v) corresponding to the Axiom of Replacement presupposes a distinction within the realm of sets. (v) would be equivalent to the tautological $\alpha \wedge b \in U_C \supset b \in U_C$ if the expression "b" was interpreted on sets in the usual sense only. "b" has to range over collections *some of which* are so that their member are in U_C, $b \subset U_C$, so that they are sets themselves. (i) – (iv) could be fulfilled by some rank V_α within V. Mac Lane distinguishes 'small' from 'large' sets. "b" in (v) can range over 'large sets'. U_C is a large set comprising *only* small sets. Therefore $U_C \notin U_C$.

Calling collections with closure conditions like (i) – (v) 'universes' invites asking for the universe of all universes (i.e. some really universal collection). In fact this is a distinction between sets *and classes* again. U_C is closer to V as to U understood here. The category of sets is a meta-category then. Applying the powerset operation to U_C yields a cardinality beyond U_C.

Category theory deals with more of these meta-categories, each dealing with a collection that is treated in other theories as a *class*. Category wants to be even more abstract, however, Mac Lane (1998, p. 23) proposing a category of *all meta-categories* or a category of proper classes. This takes up the problem that classes in **NBG** should form a collection, which cannot be dealt with in **NBG** itself.

These very large collections (e.g. the category of all meta-categories), however, play no role in theory building in (applied) category theory.

The meta-theory of category theory is usually a mixture of **FOL** and basic set theory!

If – sometimes – category theory is introduced as an alternative to a basis of mathematics in set theory (cf. Mac Lane 1998, pp.289-91) more basic concepts (like FUNCTION, DOMAIN, PULLBACKS) are needed than $\in$ in set theory. Supposedly we can easily understand them (only) because we already understand their usual sense from a set theoretic context. The concepts SET and MEMBERSHIP seem to be more intuitive and elementary. The basic axioms of set theory seem to be more natural than corresponding basic axioms of category theory as well.

In its modelling of 'small' categories category theory, therefore, can be seen as modelling up to a rank V_α in V. With respect to V and other 'large' categories, the question of the collection of them resembles the problem of the class of classes in **NBG** or **MK**.

What category theory needs can be provided by (paraconsistent) set theories with U. Such theories thus, additionally to their other merits, provide an option to unify two branches of standard mathematics, to re-integrate category theory into set theory.

III

VIRTUAL SETS AND CONSTRUCTIVISM

As the problem of universality cannot be pushed up an ontological level to classes, we have to look at set theories dealing with the presence or absence of a universal set.

Quine (1963) offers the ingenious idea of having a set theory with a universal set $U = \{x \mid x = x\}$ and not really having it at the same time!

In *Set Theory and its Logic* Quine tries to set out the common core of different conceptions of sets, i.e. he tries to develop as much set theory as possible with as little axiomatic assumptions as possible before introducing the axioms that set, say, **ZFC** and **NF** or **NBG** apart. One of his main tools in this enterprise is his theory of 'virtual' sets[34]. Virtual sets are set expressions built by curly brackets and set abstraction (like: $\{x \mid x > y \wedge x \neq z\}$) that occur on the right hand side of "$\in$". These set expressions thus are used to build statements of the form: $w \in \{x \mid \varphi(x)\}$. Since the language under consideration allows for statements like "$x \in y$" these set expressions function as singular terms syntactically on a par with variables that can be interpreted as having some set as value. The crucial point about their virtuality is that they cannot be quantified over (in that position to the right of "$\in$"). They are not members of any set (virtual or existent). Since Quine follows the methodological maxim that only those entities are admitted into a theory that are quantified over, these set expressions do not stand for or denote entities. They are short hand for statements in which conversion has occurred, i.e. $w \in \{x \mid \varphi(x)\}$ is short – depending on the length of φ, of course – for $\varphi(w)$. They are virtual also in the sense that some such expression might be quantified over *later*, so we do not know when we see such an expression whether it never materializes into a set later on. They can be quantified over indirectly in expressions like: $(\exists y)(y = \{x \mid x < 18\} \wedge y \in z)$. One can thus introduce existential commitments piecemeal. One can have true inclusion and identity statements for virtual sets without these

[34] Quine speaks of 'classes' but uses "class" synonymously to "set". Martin's theory of belief also works with virtual sets (cf. 1969, pp.123-35). (Glubrecht/Oberschelp/Todt 1983) combines ideas from the Calculus of Classes with Quine's theory of virtuality, but also *adds* virtual objects (in some kind of 'outer domain' like in Free Logic) as denotations for virtual sets!

sets existing as $\{x|\varphi(x)\} \subseteq \{x|\ \psi(x)\}$ means $(\forall x)(\varphi(x) \supset \psi(x))$; and correspondingly the identity of virtual sets is a bi-conditional statement.

Quine starts with a definition and an axiom for "=" ensuring extensionality of sets and a pair of weak axioms (providing the existence of $\varnothing$ and of pair sets, $\{x, y\}$ for all x and y), which given the framework of virtual sets provide the finite sets (only). This framework contains **FOL** (with a ι-operator) and the usual set theoretic constructions like unions and cuts. It contains the identification of objects with their unit sets![35]

Quine's framework also can express the existence of a set x by "$x \in U$" with U being the universal set $\{x \mid x = x\}$, which may itself be merely virtual, however! U only contains existents, since the "x" left to "|" carries ontological commitment. Classes are thus excluded from the theory. Existential formulas are needed, since by virtuality not every singular term refers, and the usual quantification rules have to be restricted to existing objects.

This resembles Free Logic and free usage of non-referring singular terms. Bencivenga (1976) thus tried to turn Quine's ideas into Free Set Theory (**FST**). In contrast to Quine's theory **FST** allows virtual sets to be members of virtual and of existing sets. Every virtual set has its singleton. However, as **FST** defines $\varnothing$ as the set containing no existents and postulates Extensionality as equality in *existing* members all the singletons of virtual sets are identical! All are identical to $\varnothing$. Not much is gained so. As Quine may allow for a virtual Russell Set **FST** shows the antinomic sets to be non-existent. **FST** disproves the existence of the set of all existing sets, which is only virtual in Quine's theory, but where $U \neq \varnothing$. The virtual set of all virtual sets does not exist in Quine's theory, and it is provable identical to $\varnothing$ in **FST**. So **FST** provides no real progress. [We come back to the usage of Free Logic, however, with system **APS** in chapter V.]

Virtuality is a powerful idea, as can be seen by its employment in arithmetic. Finite sets turn out to be sufficient for standard arithmetic! Each natural number can be constructed as a finite set, say the set of its predecessors (the predecessor relation being the converse of the usual successor relation). For some purposes of arithmetic we need to talk about the set of all natural numbers however. **Z** introduces the set of natural numbers for this purpose by the Axiom of Infinity. This need for infinity can be circumvented. The decisive idea is to use a virtual set instead of the Axiom of Infinity. The Axiom of Infinity uses the successor operation and so 'looks forward' towards infinity. One may also use the converse of the successor operation and 'look backwards' instead. We take $\varnothing$ as representing 0, as usual. The successor function is modelled by the function

[35] This is a substantial and controversial assumption. We come back to it later when we discuss theories for which this distinction is of outmost importance.

giving for any *x* the unit set {x}. Let us denote the predecessor function by "ϕ" and the closure of a function *f* with respect to a set *x* by "f*x". Now we can define that some number *x* is smaller or equal than a number *y* by:

(≤) $x \leq y \overset{\text{def}}{=} (\forall z)(y \in z \wedge \phi^*z \subseteq z \supset x \in z)$

i.e. x is smaller than y if x is present in all sets which contain y and are closed under the predecessor function. We can now define ℕ by

(ℕ) ℕ is short for "$\{x \mid \varnothing \leq x\}$"

Nothing demands that ℕ is more than virtual! Note that the quantifier in (≤) needs only to range over finite sets. The finite sets can be identified at this point as sets that contain some number as largest element and are closed with respect to the predecessor relation. A further axiom – a finite version of the Axiom of Replacement – is added:

(FR) The range of a function applied to a finite set exists.

This again yields only further finite sets. By this axiom mathematical induction can be derived as the scheme

(I) $\varphi(\varnothing) \wedge (\forall x)(\varphi(x) \supset \varphi(\{x\})) \wedge y \in \mathbb{N} \supset \varphi(y)$

Given the finite version of replacement, induction and the thus available notions of iteration and ancestral the well known arithmetical operations and (Peano/Dedekind) axioms for addition, multiplication and exponentiation can be derived (Quine 1963: §16).

Arithmetic can thus be done without infinity, it seems. No explicit commitment to infinity has to be introduced in the corresponding core set theory.

Quine's theory, however, gives way to ever larger infinites when the need for real numbers arises, supposing that there is a need for real numbers. Rational and real numbers are introduced as sets of sets of natural numbers. For these definitions to work (i.e. get beyond the empty set) one has to ensure that for arbitrary subsets of ℕ their union exists, and this is an existential commitment to infinity. One such axiom of infinity then is:

(INF) $(\forall x)(x \subset \mathbb{N} \supset x \in U)$.

The main problem for our topic universality is, on the one hand, again the presupposed and non explicit meta-theory. For instance: the quantifier in (≤) has to range over the set of *all* finite sets, and this set, of course, is a non-finite set. The meta-theory laying down the truth conditions for the quantifiers in this set theory has to use an infinite domain. The "x" in the definition of the universal set U has to range over all sets.

On the other hand the idea of *mere* virtuality itself seems problematic. Take the collection of all *x* such that *x* = *x*. By means of virtuality the theory can talk about that collection, but whereas the theory assumes that the items in the collection are existent (it even defines existence as membership in U) the collection is merely virtual. In case the theory develops in the direction of **Z**, the virtual set U has to stay merely virtual, on pains of deriving the antinomies using U in the Axiom of Separation (AS). In this case the treatment of universality comes down to its (non-)treatment in **Z**. In case U will be quantified over later (i.e. its virtuality is desolved into real existence) one option will a development in the direction of **NBG** U undergoing metamorphosis into a class. The treatment of universality comes down to its treatment in **NBG**, which again means its non-treatment for classes. Another option in case U will be quantified over later will be the avoidance of classes. In that case, however, we should expect some substantial changes in the set theoretic framework (e.g. exchanging Separation for a restricted subset building axiom). *Prima facie* quantifying over U makes U existent and then we have:

$$U \in U$$

contradicting the Axiom of Foundation. And we should even have – for a start! –

$$\wp(U) \in U$$

which with the immediate $\wp(U) \subseteq U$ yields more strange results. $U \in U$ not just contradicts the Axiom of Foundation, but also contradicts the standard ways of introducing cardinal or ordinal numbers (as the elements of $\wp(U)$ exist, $\wp(U)$ cannot have more members than U, thus, contradicting *Cantor's Theorem*, $\wp(U)$ has no larger cardinality than U).

All this seems plainly bizarre (of course only given our standard/iterative idea of sets). What makes this option interesting is that it deals with the problem of universality in set theory itself. We preferably explore some *set* theories *with* universal sets (i.e. universal sets which are more than virtual).

Quine's theory is embedded within standard logic and set theory. Virtual sets are virtual in the sense of not being real and not *yet* being real, but within reach of stronger axioms. Quine's standard meta-theory involves actual infinity and standard set theory.

*

Intuitionistic **ZF** (**IZF**) does not change the picture with respect to U. It even allows for a double negation interpretation of **ZF**. Some constructive set theories work with classes and introduce V as well. In this case the objections of the preceding chapter apply. Constructive and intuitionist set theories also forsake the full Powerset Axiom and may restrict Separation;

to avoid re-introducing *tertium non datur* they have to forsake the Axiom of Choice and the Axiom of Foundation (cf. Aczel/Rathjeu 2001). Thus they *add* non-intuitive features to the problem of the universal set in **Z**.

Constructivity in the sense of the Axiom of Constructability

(VL) $V = L$

with L being the realm of constructible sets (i.e. the sets which are built by separation using a formula φ of the language) makes use of the classes V and L and is in fact a non-standard version of **ZFC+GCH** (with restricted Powerset, of course).

A more radical version to the stepwise approach to set existence is (radical) constructivism. The constructivist allows only for those sets which either have been individually or generically shown to exist. Allowing for schemes of existence proofs results in embracing a *totality* where not all instances have been shown individually. Once some such large totality has been admitted larger ones result by construction. Nonetheless, as they *are* under construction even the liberal constructivist can work with the idea of a *growing* universe. The realm of mathematical objects grows as our constructive efforts enfold. Seen in this light a constructivist may hold that there is no universal set as the idea of its existence presupposes the wrong idea of an already completely present universe. (This may resemble the Kantian undermining of 'the antinomies of pure reason', which each presuppose – according to Kant illegitimately – a developing series as also completely 'given'.)

One set of objections to this constructivism focuses on issues of cardinality. The sets we (as human kind) have constructed individually are only finite. Generic proofs may put an infinity of sets within reach. Again the number of thee proof schemes we (as human kind) have constructed will be finite. The constructivist's position seems to fall back into a theory of a merely *potential* infinite or even *strict finitism* (of no infinity at all). Both positions deviate substantially from received, successful mathematics. They carry the burden of proof whether they can deliver what the sciences need. Strict finitism may commit us, further on, to paraconsistency (cf. Bremer 2007).

The second set of objections focuses on the *current stage* of construction. Even if construction work continues, at every single stage of construction we may ask whether there is a universal set containing all the sets constructed so far. *At any stage*, shouldn't $\{x \mid x = x\}$ exist? All questions concerning U thus return, even if they are now aimed at a succession of ever increasing universal sets. Non-realism so does not help at each step.

This obviously applies to a predicative set theory of rank-wise construction of sets within an iterative hierarchy (cf. Wang 1970,pp. 559-623).

IV

CONSISTENT SET THEORIES WITH UNIVERSAL SET

On occasion of the 1971 Berkeley symposium celebrating Alfred Tarski's achievements in logic and algebra Alonzo Church, who otherwise did not work much in set theory, presented a new system of set theory (Church 1974).

Church saw the two basic assumptions of post-naïve set theories in a restriction of comprehension to a form of separation (as in **ZFC**) and in a limitation of size (as in **NBG**). Similar to the criticism levelled against Limitation of Size in chapter II above Church regarded Limitation of Size as *ad hoc* (against the antinomies) and 'never well supported' as it proclaims a stopping point of further structures although classes are introduced (in **NBG** and **MK**) as collections, which can be quantified over. Church's set theory – let us call it "**CST**" here – follows **ZFC** in its idea of separation, but allows for collections that are 'large' in a way that even the larger transfinite sets of **ZFC** are not. **CST** does *not* introduce classes, but introduces a distinction *within* the area of sets. It allows even for U = {x | x=x}.

CST distinguishes between 'low sets', which have a 1:1-relation to a well-founded set, 'high sets', which are (absolute) complements of low sets and 'intermediate' sets which are neither. These labels pertain to the cardinality of sets. High sets are in 1:1-correspondence to the universal set U, low sets never. Because of the **CST** version of the Axiom of Choice for a set *x* which is not low *every* ordinal has a 1:1-relation to some subset of *x*.

So the universe of **CST** may consist only of sets, but not all are well-founded. Obviously U∈U. U is the complement of ∅, so U is the paradigmatic high set. ∅ is well-founded, and, of course, ∅∈U. The constant predicate "wf( )" expresses the property of BEING WELL-FOUNDED, defined in the usual sense (using some order relation "<"):

$$(\forall x)(wf(x) \equiv x=\emptyset \vee (\forall y)(y \subseteq x \supset (y \neq \emptyset \supset (\exists z)(z\in y \wedge (\forall w)(w\in y \supset z < w))))).$$

CST can be phrased as a second order system, quantifying over single- or two-argument open formula φ. One could understand this second order quantification as using classes, but only given a full-blown second order semantics [cf. Chap. II]. One could use schemata instead (i.e. having only free variables for open formula), as in **ZF**. We follow the first option here and make **CST** a second order theory.[36]

The **CST** axioms are:

- Extensionality, Pair Set, Sum Set, Infinity as in **Z**
- Choice: $(\forall R)((\forall x)(\exists y)R(x,y) \supset (\exists f)(\forall x)R(x,f(y)))$
- Product Set: $(\forall y,z)(y\in z \supset (\exists u)(\forall x)(x\in u \equiv y\in z \supset x\in y))$
- Separation, Powerset, Replacement *restricted* to a condition "wf(x)",

e.g. Axiom of Powerset: $(\forall x)(wf(x)\supset(\exists y)(\forall z)(z\in y \equiv z \subseteq x))$,

Axiom of Separation: $(\forall x,F)(wf(x)\supset(\exists y)(\forall z)(z\in y\equiv z\in x \wedge F(z))$ [where "y" is not free in "F"]

The Axiom of Product Set allows having a substitute for separation in high sets. What is missing is the Axiom of Foundation of **ZF.**

These axioms of **CST** are strong enough to yield **ZF**. Dropping the non well-founded sets one gains a **ZF** universe. The two theories are equi-consistent (cf. Church 1974, §5). And without violating this relative consistency **CST** can be extended by axioms which go beyond **ZFC**. These are: Strong Choice (that U can be well-ordered), Cardinality Axioms (that there are cardinal numbers in the sense of Frege and Russell for all well-founded sets) and especially the Axiom of Complements:

$(\forall x)(\exists y)(\forall z)(z\in y\equiv z\notin x)$.

The existence of absolute complements and the existence of U make **CST** a more natural set theory than **ZFC**, one may argue.

The argument in **ZFC** from the Axiom of Separation to the non-existence of U and *Cantor's Theorem* pose no problem for U and $\wp(U)$ for the same reason: the Axiom of Separation and the Axiom of Powerset are restricted to well-founded sets.

The argument to the non-existence of U [cf. Chap. I] now establishes that U *is not a well-founded set*. We knew that before. As U is not well-founded we do *not have* $\wp(U)$ in the first place.

One might now argue: So, in **CST** as well, there are some collections which are *there* – in this case *inter alia* the collection of all subsets of U, which are obviously existent if U is – but cannot be collected into a set; some collections which should exist, like $\wp(U)$, do not exist, because they

[36] Remember that in a full blown second order setting the Axiom of Choice is *not* equivalent to the Well-Order Principle. In fact the Well-Order Principle is *not* a theorem of **ZFC2** (cf. Shapiro 1991, pp.106-108). Neither the presence of the Axiom of Choice nor the presence of Foundation implies that all sets can be well-ordered in **ZFC2**.

are too large, just as in **NBG** set building operations cannot be applied to classes. Like **NBG** and **MK** tell us not much about classes, **CST** uses 'large' sets, but large sets cannot do much, since they are not subject to Separation or other set building principles.

This criticism, however, should be kept apart from a similar criticism levelled against **NBG**. In fact, all subsets of U are collected into *a set* – U itself. What cannot be done is separating a set $\wp$(U) from U. The same applies to the ordinals: U is *a set* in which all ordinals are collected, but – on pains of re-introducing the Burali-Forti antinomy – we cannot separate a set which collects only the ordinals. So does **CST** make some progress in comparison to **NBG**? On the one hand **CST** can avoid using the second ontological category of classes, with all its problems. On the other hand **NBG** can collect just the ordinals into *their own* proper collection, albeit a class. **NBG** cannot introduce $\wp$(V), as **CST** cannot introduce $\wp$(U).

It is true that although the principal idea behind **CST** was rejecting Limitation of Size, **CST** exhibits some shadow of Limitation of Size: All high sets are by definition of the same size as U, just as in **NBG** all classes are of the same size as the class of sets.

*

The widest known set theory with a universal set U – for which we have U∈U – is Quine's **NF** (from his paper "New Foundations of Mathematics", 1937). **NF** is Quine's set theory with a universal set that is not just virtual is **NFU**.

NF works by Extensionality and a Comprehension Scheme that is not as restricted as in **ZFC**.

Quine's **NF** Comprehension Scheme uses the idea of *stratified formula* (similar to the simple theory of types):

$$(\exists y)(\forall x)(x \in y \equiv \varphi(x))$$

where "y" is not free in φ and φ is stratified. A formula is stratified if the set on the right hand side of "∈" is of a higher level than that on the left, and its definition does not include that on the left. A test for stratification consists in level assignments for sets or in trying a translation into the simple theory of types. The language of **NF** itself is not typed, thus avoiding duplication of structurally identical sets at different levels, but **NF** uses the stratification test to avoid the antinomies. **NF** is equiconsistent with the simple theory of types supplemented with the full ambiguity scheme, which asserts that a formula φ is equivalent to formulas φ^1, φ^2 … structurally similar to φ with type levels uniformly raised by 1, 2 …

NF has the power to introduce Pairing, (Absolute) Complement, Powerset, and Union as instances of Comprehension.

NF allows U = {x | x = x}, since "x = x" is a stratified formula. {x|x∈y} can be stratified, defining the so-called 'essence' of an object (the collection of all its properties). "|x| = |{∅,{∅}}|" is stratified as well, so natural numbers (in this case: 2) can be understood in Frege's way. Stratification excludes, however, the usual definition of an infinite set (as in the Axiom of Infinity in **Z**). One meets in proofs and constructions in **ZFC** many unstratified set definitions. **NF** has to forsake these sets or has to introduce workarounds. Cardinal numbers and ordinal numbers come apart (cardinal numbers are not special ordinal numbers as in **ZFC**).

Non-stratified formula can be *used* in **NF** (this is different to Type Theory), but they cannot be used to define sets. Since non-stratified formulas can be used in **NF** one does not need a universal set or an empty set for every level (as in Russell's Type Theory) to have well-formed formula.

The antinomies – especially the (original) Russell Set – are avoided, since the corresponding open formulas in the Comprehension Scheme are not stratified. **NF** itself is not known to be consistent. **NF** with the Axiom of Counting, which says that a cardinal number is equal in cardinality to its singleton image, can prove the consistency of **Z**. No relative consistency proofs to **ZF** are available. Note that instances of Replacement are not stratified. Some subsystems of **NF** have been shown to be consistent (cf. Forster 1992, Holmes 1999).

U∈U means that *Cantor's Theorem* does not hold (in general) in **NF**; but the set of unit sets of its elements is smaller in cardinality than U itself! The usual proof of *Cantor's Theorem* defines a set y∈℘(x) relative to a supposed bijection *f* between *x* and ℘(x) as y = {z|z∈x ∧ z∉*f*(z)}, which is not stratified. In **NF** one can define, however, y = {z|z∈x ∧ z∉*f*({z})}, which is stratified with *f* being a supposed bijection between the set of singletons of z∈x and ℘(x). By the usual indirect argument one sees that there is no bijection between {z|z={w}∧w∈x} and ℘(x), which means |℘(x)| > |{z|z={w}∧w∈x}| = |$℘_1$(x)|. As ℘(U) ⊆ U we have |U| ≥ |℘(U)| and thus with the previous inequality we know that the cardinality of the set of singletons of elements of U *is smaller* than |U|! |U| > |$℘_1$(x)|. That could mean that not all elements of U possess a singleton: although we have the set of all singletons in **NF**, as {x| (∃y)(x = {y})} is stratified, not all objects seem to have a singleton. This cannot be the solution. Comprehension provides a singleton for any object *z*: "x∈y ≡ x = z" can be stratified. Contradiction is avoided finally by the *non-existence of the function* which maps any object to its singleton. This function does not exist *even though* every object has its singleton!

NF has other highly controversial features like the existence of infinite descending chains of cardinals, what conflicts with the Axiom of Choice, which thus doesn't hold in **NF**, which again implies – even in the absence of an Axiom of Infinity – that the universe of **NF** has to be infinite, since all finite sets can be well-ordered. The universe of **NF**, supposedly U, cannot be well-ordered then. **NF** might be consistent, though, with the claim that all well-founded sets can be well-ordered. Some functions (like the successor function) are *not* part of the universe – so where are they?

One can extend **NF** by introducing classes. In fact if one *denies* in **NF** Rosser's Axiom of Counting [see above] one can prove the existence of non-set collections which are finite! Indeed a 'strange landscape' (Forster 1992, pp.29-32).

If one restricts the set building axioms to sets and uses unrestricted class comprehension one arrives at Quine's system **ML** (cf. Quine 1963, §§40-42). The problems the system **NF** has with unstratified induction and its incompatibility with the Axiom of Choice are resolved then. There is a class of all sets, ∪U, and *Cantor's Theorem* does not apply to it, as it is no set (i.e. has no powerset at all). Apart from resolving these problems classes play no constructive role in **ML**. Obviously we find ourselves in a system very similar to **MK**, and corresponding criticism applies here [cf. Chap. II].

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A version of **NF** that tries to avoid many of the peculiarities of **NF** is **NFU** (**NF** with urelements), developed by Randall Holmes (2005). **NFU** is built from **NF** by adding urelements and restricting extensionality to non-empty sets, introducing ∅ by an axiom. One may think of it as a subsystem of **NF** that allows only such models which contain the urelements.

NFU is consistent! It is consistent with the Axiom of Choice!

NFU is an extremely strong set theory. It can provide models for **ZFCU** by having *very* large cardinals (strongly inaccessible cardinals).

By working with a longer (*finite*) list of simple axioms (like Extensionality, Complements, Unions, Singletons, Cartesian Products, Converses, Domains, Projections, Singleton Image of a Set, Choice (!), Infinity…) Stratified Comprehension can be proven as a theorem!

The universal set is provided by its own axiom:

(U) $\{x| x = x\}$ exists.

U contains all sets as elements. All sets can be well-ordered, which is equivalent to the Axiom of Choice. So U can be well-ordered, in contrast to

U in **NF**. By the existence of U, the Axiom of Complements and Stratified Comprehension absolute complements exist.

NFU also has its peculiarities: urelements, atoms and ordered pairs (!) are taken *as primitive*. One wonders what is supposed to be 'in' the ordered pair $<x,y>$ if not *x* and *y*. *And* the **NFU** Axiom of Ordered Pairs contains the standard identity condition on ordered pairs *without* saying that an ordered pair is a set. The Axiom of Pairing of **Z** looks more natural.

More problematically: $[\in]$, $\{<x,y> \mid x\in y\}$, does *not* exist, the existence of $[\in]$ leading to antinomies.[37] Whereas we thus have a set U, which may stand in as the extension of the predicate "set", we have no extension for the membership relation, although $[\in]$ looks as natural as U. Interestingly $[\subseteq] = \{<x,y> \mid x \subseteq y\}$ exists. So in many cases "$\{x\}\subseteq y$" may substitute for "$x\in y$".

The non-existence of $[\in]$, which, of course, entails $[\in]\notin U$, entails that the membership relation is not modelled by U (U has no element corresponding to it), so that U cannot be a model for **NFU** itself. So as with the standard set theories **ZF**, **ZFC**, although working with a universal set, **NFU** has to look *outside of itself* for models. If the universal set is really *universal*, where should that outside be? We seem to be back to larger cardinals or similar collection like entities or some hierarchy [cf. Chap. I & II].

Not every supposed set exists, e.g. the set of all Cantorian ordinals. This is not better than in **ZF**. Some collections (like U) which do not exist in **ZFC** can exist in **NFU**, but as **NFU** does not distinguish sets from classes, some collections (like that of Cantorian ordinals) which exist in **NBG** do not exist in **NFU**. **NFU** can be viewed as trying to capture some middle ground between the other systems. This yields its own peculiarities.

Especially problematic is that some version of *Cantor's Theorem* is provable. The large sets (like the set of ordinals or U) in **NFU** have the strange property of *not* being equinumerous to their singleton images! How can that be? Although every object *has* a singleton (by the Axiom of Singletons), just as in **NF** the *function* giving the singleton to every object does not exist.[38]

$|\wp_1(U)| < |U|$ seems to contradict the Axiom of Singletons, a *provable* contradiction *only* being avoided by the non-existence of a general

[37] *Proof* (Outline). If $[\in]$ exists, so does its complement $-[\in]$ by the Axiom of Complements. $[=] = \{<x,y> \mid x = y\}$ exists, since "$x = y$" is stratified. Then the cut of $[=]$ with $-[\in]$ exists, and this cut is a cousin of the Russell Set: $\{<x,y> \mid x = y \wedge x \notin y\}$.

[38] *Proof* (Outline). If the function f:$U\rightarrow\wp_1(U)$ existed, extensionality of the singleton would yield a function f^{-1}, so $|U| \leq |\wp_1(U)|$, which contradicts the Cantorian argument to $|\wp_1(U)| < |U|$ (given above with respect to **NF**). ■

Holmes (2005, pp. 109-110) provides another proof which relates to stratification.

singleton function (cf. Quine 1963, p.293). We have – so to say, in analogy to Separation restricting Comprehension in **Z** – restricted singleton construction!

So a corollary of the non-existence of the (general) singleton function is *that the cardinality of the set of singletons of members of U,* $|\wp_1(U)|$, *is less than the cardinality of U*. This leads in **NFU** to the distinction between 'Cantorian sets' *x* with $|x|=|\wp_1(x)|$ and 'non-Cantorian sets', which resembles other limitation of size distinctions.

These cardinality issues lead to *Specker's Theorem* (cf. Holmes 2005, pp.132-34):

$$|\wp(U)| < |U|$$

which is read as proof that there are (many, many) urelements/atoms besides sets in U. Atoms, which have no members, are not elements of $\wp$(U), which contains all subsets of U, but not non-sets like atoms. If there are no atoms one expects for a system with a universal set U that one has $|\wp(U)|=|U|$. In fact *most* objects in U in **NFU** then have to be atoms or pairs (i.e. ordered pairs not reducible to sets).

Unlike some version of **NF** in (Quine 1963) atoms are *not* identified with their singletons in **NFU**. As mentioned, ordered pairs are also objects in their own right besides sets. Ordered pairs can be taken as atoms in **NFU** as no claim with respect to composition – only with respect to their identity condition – was made. *Specker's Theorem* mirrors this conception.

Again:

> *Specker's Theorem* for **NFU** asserts that most entities in U are not subsets of U, which means *most* of the universe has to consist of urelements!

So: in all known models of **NFU** $|U| > |\wp(U)|$! All relations are subsets of U×U, and all functions f:U→U should, if they are allowed to exist, themselves be sets of ordered pairs, thus be elements of U, thus be available as their own arguments, thus sometimes be forbidden to exist *as sets* at all in a consistent setting (e.g. a function of negative self-application). **NFU** itself can have models only in a realm which possesses properties quite different from what we expect of sets. **NFU**, so, deals not just with sets. In fact the non-sets *vastly* outnumber the sets in any model of **NFU**. Ideally the non-sets contain just the urelements, but by the argument above concerning the Axiom of Singletons we should *expect* there to be a collection containing the ordered pair of any *x* and its singleton, but this collection cannot be a set in **NFU**. A crucial question is whether **NFU** can at least recapture ordinary sets – **ZFC**-like entities – as a sub-domain. Even though this is possible, however, we regain simply **ZFC** as a sub-universe – and are none the wiser with respect to our universality problem, as U, because of its behaviour in **NFU**, cannot be part of that recaptured realm. U is not ‘Cantorian’. Even the ordinals of **NFU** cannot be well-ordered in a *set model* of **NFU**! Holmes does not introduce proper classes into **NFU**, but admits their existence, supposedly objects of a broader theory (cf. Holmes 2005, p.50). **NFU** is not truly universal, as well.

NFU, thus, may provide a lot of machinery to do ordinary mathematics, the gain with respect to our set theoretic intuitions brought by having a universal set U, however, seems to be more than lost by the consequences of *Specker’s Theorem* and the absence of [∈].

V

Paraconsistent Set Theories with a Universal Set

ZFC, **NF/U**, even **NBG** and **MK** forbid the existence of collections which intuitively should be there as all items to be collected are there: be it not just U, but the set of all ordinals, [∈], the general singleton function, the pair of the class of ordinals and the class of cardinals etc. Maybe these collections are not sets, set theory only dealing with sets. Mathematics, maybe, has no practical use for other collections. Maybe – maybe not. If some theory saves the intuition that these collections are collectible this could furnish it with a crucial advantage over its competitors.

In **Z** there is neither an unrestricted comprehension axiom or schema nor a universal set. Paraconsistent set theory regains both.[39]

Apart from semantic closure set theory is one of the main motivations for the strong paraconsistent approach (so-called 'dialetheism'), which accepts both that there are some true contradictions as well as the existence of inconsistent objects. The consideration starts with the simple question: What is a set?

[39] Whereas in the other paragraphs common knowledge of standard logic was assumed we have to divert in this paragraph several times to outline the basics of the involved paraconsistent logics or theories. In most cases only a rough sketch is provided to save space. There are excellent introductions to Relevant and Paraconsistent Logic (and paraconsistency in general) on the market. Brady (2006), Priest (2006) and Routley (1980) supply in-depth coverage of their respective systems, semantics and theories. The preferred system in this chapter, **APS**, is outlined in somewhat greater detail, as it cannot be found elsewhere. Therefore in this chapter additional *schema* are used which show the syntactic type of expressions more clearly: "á", "é" are a schema for individual terms, "P" is schematic for some general term, "R" for some relation. "A", "B"…(sentences), "G( )", "F( )" … (general terms), "a", "b"… (singular terms) are *abbreviated* expressions of the formal languages themselves. We allow rules to use these expressions. In case rules or axioms involve no schemata but abbreviations the systems are understood as containing rules of uniform substitution into an appropriate syntactic type, excluding, of course, substituting into logical constants (like "=" or the existence predicate "E!( )").

The standard account of concepts in **FOL** semantics goes like this: What does “( ) is a tree” refer to? It refers to the set of all trees. A concept/property is understood if we understand what its extension is. Now, what is a set? It cannot be the extension of “( ) is a set”, since this extension would be a universal set, but there is none in **Z**, the standard set theory. So in standard meta-theory there is no set/extension corresponding to our usage of “( ) is a set”. For restricted usages (sets of some kind, cardinality, order etc.) there are sets, but one cannot speak of sets in general. Standard set theory seems using a fundamental notion that can at best be partially explained *by this theory*! This runs against our intuitive understanding of “set”.

Naïve Comprehension expresses not just naivety, but the intuitive idea of collecting objects with respect to some condition or property. Especially “x=x” looks innocent enough to warrant a collection.

And the absence of a universal set is not just a problem of understanding what a set is. Some set theoretical explanation of other concepts make use of a universal quantification about sets (cf. Priest 2006, pp.28-37).

If one defines:

> $\Sigma \vDash \varphi \stackrel{\text{def}}{=} \varphi$ follows from a set of premises Σ if and only if every interpretation that makes all $\gamma \in \Sigma$ true makes φ true.

one talks about *any* interpretation. And the domain of an interpretation is arbitrary. It may be a set of arbitrary high rank. So the supposed definition talks about *all* sets of an arbitrary high rank (i.e. of the completed hierarchy), but *in* **ZFC** we can never get at *all* sets unified! So it seems that our understanding of consequence cannot be modelled by **ZFC**. **ZFC** can only define an incomplete model thereof.

And if there is no universal set, there is no universal complement of a set. Some theories (category theory) want to talk about such sets, however, as we have seen. Granting this reasoning some *prima facie* plausibility calls for a closer look at paraconsistent set theories. If the costs of paraconsistent set theories add up more than expected, some of the supposedly ‘intuitive’ and ‘innocent’ claims have to be reconsidered [in chapter VI].

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Richard Routley was one of the first to introduce. paraconsistent set theory. He uses the Relevant Logic **DL** (cf. Routley/Meyer 1976)

Axioms:

(A1) $A \to A$

(A2) $(A \to B) \wedge (B \to C) \to (A \to C)$

(A3) $A \wedge B \to A$

(A4) $A \wedge B \to B$

(A5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A6) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A7) $\neg \neg A \to A$

(A8) $(A \to \neg B) \to (B \to \neg A)$

(A9) $A \to A \vee B$

(A10) $B \to A \vee B$

(A11) $(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$

(A12) $\neg A \wedge \neg B \to \neg(A \vee B)$

(A13) $\neg(A \wedge B) \to \neg A \vee \neg B$

Rules: (R1) $\vdash\varphi, \vdash\varphi \to \psi \Rightarrow \vdash\psi$

(R2) $\vdash\varphi \to \psi \Rightarrow \vdash(\neg\psi \to \neg\varphi)$

with the following quantificational extension (Routley 1980, p.290):

Axiom schema:

(A14) $(\forall x)P(x) \to P(á)$

(A15) $(\forall x)(A \to P(x)) \to (A \to (\forall x)P(x))$ *

(A16) $(\forall x)(A \vee P(x)) \to (A \vee (\forall x)P(x))$ *

(A17) $(\forall x)(P(x) \to A) \to ((\exists x)P(x) \to A)$ *

[* x not free in A]

Rules: (R3) $\vdash\varphi \Rightarrow \vdash(\forall x)\varphi$

(Naïve) Comprehension is expressed with a Relevant conditional:

(NC_R) $(\exists y)(\forall x)(x \in y \leftrightarrow P(x))$

(NC_R) has no restrictions on "P( )" (like "y does not occur in P( )") so that one can have a set y such that $x \in y \leftrightarrow \neg x \in y$ (taking "¬( )∈ y" as "P( )"). This is a bizarre set of all things that belong to it iff they do not belong to it!

Such usages of (NC_R) immediately raise the question of inconsistent ontology (i.e. *whether* and *where* are objects like the set described?).

(NC_R) is no longer well-founded: a set like the Routley Set $y = \{x \mid x \in y \leftrightarrow \neg x \in y\}$ may contain y itself. (NC_R) allows for $\{x \mid x \in x\}$.[40] There is even the set $y' = \{y \mid x \in y \leftrightarrow \neg x \in y\}$ – etc.

Rephrasing the Axiom of Extensionality using a relevant biconditional allows deriving even the Axiom of Choice from it and (NC_R). Since also irrelevant theorems follow, the Axiom of Extensionality is replaced by a rule and a definition:

(EXT_R) (i) $x = y \Rightarrow x \in z \rightarrow y \in z$
(ii) $x = y \overset{\text{def}}{=} (\forall z)(z \in x \leftrightarrow z \in y)$

This rules allow for the substitution of identicals and defines how identity of sets is to be taken. For substitution we have:

(SUB) $\vdash A \leftrightarrow B \Rightarrow \vdash \varphi(A) \rightarrow \varphi(B)$

i.e. if A and B relevantly imply each other than B can be substituted in any context φ of A for A so that the resulting sentence is still relevantly implied. (Different antinomies do not imply each other.)

(NC_R) allows defining sets otherwise introduced by axioms:

Existence of the empty set ∅:

$(\exists y)(\forall x)(x \in y \leftrightarrow \neg x{=}x)$

This set is empty, since even in **DL** we have: $(\forall x)x{=}x$

Existence of the *absolute* complement of some set x:

$(\exists y)(\forall z)(z \in y \leftrightarrow \neg z \in x)$

x is some arbitrary set here, so that we can have the complement of any set we wish. In **ZFC** there are only relative complements of *x* (in some superset) because of the more restricted Axiom of Separation.

The antinomies can be derived in this set theory, but the underlying paraconsistent logic avoids triviality.

That versions of the Axiom of Choice can be derived within his set theory shows, according to Routley, the *realistic* character of paraconsistent set theory (i.e. the domain is simply there, with all functions defined on it, whether we have constructed them from previously constructed material or not). If *w* is any family of non-empty (disjunct) sets *v*, any set theory with

[40] So the Foundation Axiom of **ZFC** is not part of paraconsistent set theory based on unrestricted comprehension. Foundation was a late comer in **ZFC** any way, has no mathematical applications outside set theory, and is dropped in otherwise standard non-founded set theory (cf. Aczel 1988).

unrestricted Naïve Comprehension (like Routley's theory here, but also like **APS** later in this chapter) can circularly introduce the choice function

$$c_w = \{<v,x>| \, v\in w \wedge x\in v \wedge \neg(\exists z\in v)(z \neq x \wedge <v,z>\in c_w)\}$$

The anti-foundationalist universe can be well-ordered then.

What about the metalogic of Routley's set theory? Note that Routley's theory is a set theory employing neither many-sorted variables nor working with classes. Routley (1980, pp.931-33) extends his logic **DKQ** by some axioms for arithmetic to his paraconsistent arithmetic **DKA**. He can prove that **DKA** is not trivial, i.e. absolute consistent: $(\exists\varphi)\nvdash_{DKA}\varphi$. A system like **DKA** being inconsistent does not meet the condition of *Gödel's Second Theorem*, so can be used itself to prove its own (absolute) consistency. This proof by Routley, however, uses a truth functional conditional like that of the paraconsistent logic **LP**. So this proof – because of the Curry Conditions (i.e. conditions allowing deriving a version of *Curry's Paradox*) – cannot be extended to paraconsistent set theory.[41] Routley and Brady (1989) nevertheless proved the non-triviality of a paraconsistent set theory using a logic with a negation semantics in terms of the Routley star * and the ternary accessibility relation, which are both highly controversial in being considered artificial by many. Brady improved on that situation by proving the non-triviality of an inconsistent set theory (i.e. one involving inconsistent sets) with respect to a truth-functional dialethical semantics (cf. Brady 2006, pp.242-45), the matrixes of which, however, are contrived to the purpose and not as natural as the matrixes of **LP**. He states his set and class theory in his logic **DJ^{d}Q.**

Relevant Logic based set theories like Routley's, however, *violate* the idea that sets are extensional. The relevant conditional "→" is intensional (usually having a possible worlds semantics). By its use in (NC) sets become intensional! Limitations of substitutivity with "→" carry over to sets. For instance $x\in y \leftrightarrow x\in y \wedge \alpha$ (for some truth α) does not relevantly hold true, thus y and $\{x \mid x\in y \wedge \alpha\}$ *although having the same members* cannot be said to be identical (cf. Priest 2006, pp.253-55).

*

[41] On **LP** and the Curry Conditions cf. Priest 1987, 2006, Restall 2000, Bremer 2005.

Brady's version of paraconsistent set theory (cf. Brady 2006) also employs the ideas behind (NC_R) and (EXT_R). The underlying logic is $\mathbf{DJ^dQ}$.

Axiom schemes:

(A1)	$A \rightarrow A$
(A2)	$A \wedge B \rightarrow A$
(A3)	$A \wedge B \rightarrow B$
(A4)	$(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(A5)	$A \rightarrow A \vee B$
(A6)	$B \rightarrow A \vee B$
(A7)	$(A \rightarrow B) \wedge (C \rightarrow B) \rightarrow (A \vee C \rightarrow B)$
(A8)	$A \wedge (B \vee C) \rightarrow A \wedge B \vee A \wedge C$
(A9)	$\neg\neg A \rightarrow A$
(A10)	$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
(A11)	$(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
(A12)	$(\forall x)P(x) \rightarrow P(á)$
(A13)	$(\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x))$ *
(A14)	$(\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x))$ *
(A16)	$P(á) \rightarrow (\exists x)P(x)$
(A17)	$(\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A)$ *
(A18)	$A \wedge (\exists x)P(x) \rightarrow (\exists x)(P(x) \wedge A)$ *

* [x not free in A]

Rules:
(R1) $\vdash\varphi \rightarrow \psi, \vdash\varphi \Rightarrow \vdash\psi$
(R2) $\vdash\varphi, \vdash\psi \Rightarrow \vdash\varphi \wedge \psi$
(R3) $\vdash\varphi \rightarrow \psi, \vdash\gamma \rightarrow \delta \Rightarrow \vdash (\psi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \delta)$
(R4) $\vdash\varphi \Rightarrow \vdash(\forall x)\varphi$

Meta-Rules:
(MR1) If $\vdash\varphi \Rightarrow \vdash\psi$ then also $\vdash\varphi \vee \gamma \Rightarrow \vdash\psi \vee \gamma$
(MR2) If $\vdash\varphi \Rightarrow \vdash\psi$ then $\vdash (\exists x)\varphi \Rightarrow \vdash (\exists x)\psi$

where in both meta-rules in the derivation $\vdash\varphi \Rightarrow \vdash\psi$ (R4) does not generalize on a free variable in φ.

Brady invented a semantics of content containment to avoid the unnatural possible worlds semantics common to Routley's systems. The content of φ comprises everything that can be 'analytically established' from φ (i.e. with respect to the meaning of φ). Given the semantics of *content containment*

Brady has to use (EXT_R), since the *content* of x = y seems not to *contain* $x \in z \leftrightarrow y \in z$.

In fact Brady's theory is foremost a *class theory* (not a set theory). Brady distinguishes classes for which the two axioms hold and for which sentences dealing with them have a relevant logic from sets, whose membership sentences obey standard logic! The standard behaviour is needed to have enough countable sets in the classes. And not using "$\rightarrow$" for sets keeps *them* extensional. The classes, however, are intensional and have other identity conditions than sets. Classes, which also comprehend sets, are thus in at least two respects quite different from sets. Classes comprehend individuals, sets and classes having a property. Brady proposes several comprehension schema like

$$x \in \{yY \mid \varphi\} \leftrightarrow \varphi^{x/y\ \{yY|\varphi\}/Y}$$

"y" being free for "x" and φ *maybe* having a free class variable thus speaking about its corresponding class *itself*. Classes are 'logical' collections, sets are arbitrarily formed well-founded collections. Because of their different logical behaviour the null *set* and the null *class* have to be different. Like in **NBG** some classes correspond to a set (are 'classically identical' to a set, cf. Brady 2006, p.183, 311). Not all properties built sets, only those with 'classical membership statements'. The collection of well-founded sets, for instance, cannot be a *set* itself. Brady's theory thus consists of two parallel sub-theories: one for sets and one for classes. Ordered pairs are – like in **NF** – taken as primitive as well! Numbers are also distinct, since they are not reduced to sets!

Brady has proved his system of set and class theory to be non-trivial and even consistent in the narrow sense of not $\vdash\varphi$ and $\vdash(\neg\varphi)$, but possibly $\vdash(\varphi \equiv \neg\varphi)$, on the condition that large parts of **ZF** are consistent.

Brady's version of paraconsistent set theory does not contain all of the antinomies and 'only' keeps them from spreading triviality elsewhere. Some of the antinomies do not occur. In case of the Russell set one can prove $R \in R \leftrightarrow R \notin R$. To get to the explicit contradiction $R \in R \wedge R \notin R$ once needs either the Law of the Excluded Middle or Negation Introduction. Both are absent in $\mathbf{DJ^dQ}$. Thus given the validity of $R \in R \leftrightarrow R \notin R$ only one can chose to make them both true or both false. Something similar holds for *Curry's Paradox*, since Contraction does not hold in $\mathbf{DJ^dQ}$. Brady works by rejecting Excluded Middle and claims that this is not *ad hoc* to avoid the antinomies like $R \in R \wedge R \notin R$. He argues, however (cf. Brady 2006, pp.40-41), for the rejection of Excluded Middle by *reduction* starting from the observation that otherwise antinomies were provable. Comparing restricted comprehension (like Separation) or restricting negation (in giving up Excluded Middle) one may well argue that NEGATION is an even more central concept and not to be messed with lightly. In fact Brady rests his case on his logic $\mathbf{DJ^dQ}$ in which Excluded Middle fails and negation is taken as an *intensional* connective/operation. The majority of logicians doubt that, as we seem to understand negation naturally in terms of truth, not meaning or content. Brady's approach thus leads to the broader topic of negation, which cannot be taken up here. The burden of proof of building a viable set theory only by messing with the extensionality of negation rests with accounts like Brady's.

Since Brady distinguishes sets from classes he restricts the validity of *Cantor's Theorem* to sets, avoiding the antinomy that the powerset of the universal set has to be within the universal set and at the same time larger than the universal set. The sets are collected into a class (cf. p.301). After all Brady's theory turns out to be similar to theories like **MK** or Ackermann's set theory [cf. chapter II]. The collection of all sets is a class. Some classes are even within sets (are members), but these are only the classes corresponding 'classically' (i.e. in standard logic) to sets. Classes are not comprehended into a universal class of all classes. Brady's theory thus does not make progress in comparison to those theories with respect to our quest for a truly universal collection of all collections, or at least a set of all sets.

*

One may try to gain both a paraconsistent treatment of antinomies and a substantial amount of 'classical recapture' by adopting an adaptive logic, extending it with the two basic set theoretical rules. Ideally the resulting set theoretical logic should combine the basic power and many of the results of straightforward paraconsistent set theories (like Routley's or Brady's) with a severe restriction on reasoning with or multiplying inconsistent objects. It should avoid classes.

The base logic may be the adaptive version of **LP** (Priest's so called "Minimal Inconsistent LP", 1991) with standard quantificational extensions giving **ALPQ** (**A**daptive **LPQ**). Since set theory needs identity one in fact needs $\mathbf{ALPQ^=}$. $\mathbf{LPQ^=}$ however has a too weak concept of identity, so we need some restrictable but more powerful rule.

We have to introduce some basic ideas of adaptivity first. Adaptive logics (and proofs) derive consequences from a premise set, but are adaptive in that they retract some consequences if their derivation crucially depended on the inconsistency of the premise set.

Retracting in the process of reasoning from a premise set cannot be completely avoided, since there is no general algorithmic procedure (for just any logic) to test whether Γ or $\Gamma \cup \{\varphi\}$ is consistent. So we often extend our premise set Γ by a new assumption on the supposition that this extension is consistent, although it sometimes turns out not to be. Especially if $\Gamma \vdash \varphi$ *depends* on $\Gamma \nvdash \psi$ and no negative test is available for $\Gamma \nvdash \psi$, then we have even no positive test for $\Gamma \vdash \varphi$. (The derivability of φ may depend on the absence of ψ if say ψ states some exceptional condition on employing some rule to derive φ.) Retraction is of most interest with respect to internal dynamics, since given one and the same premise set the sentence φ may be derivable at some stage and retracted later. φ then might not belong to the final consequence set, but it appeared to during some stages of the reasoning process. An adaptive logic is characterized by two logics:

(a) The *Upper Limit Logic* (ULL) allows for the unrestricted application of logical rules to derive the most consequences possible. Typically ULL is (standard) FOL;

(b) The *Lower Limit Logic* (LLL) is chosen to model some type of restricted reasoning. In our case it is a paraconsistent logic, i.e. a logic that blocks the application of some rules of standard logic.

The *adaptive strategy* is the way to handle the management of restrictions and the corresponding retractions. An adaptive logic generates a set of consequences of an (inconsistent) Π that can lie between $\mathrm{Con}_{\mathrm{LLL}}(\Pi)$ and $\mathrm{Con}_{\mathrm{ULL}}(\Pi)$. The idea of adaptation is therefore: Think of some critical rules as applicable and make exceptions *only* if one of the premises is known to be inconsistent (or problematic in some other ways to be explained below). Since we do not know beforehand which premises *are* consistent, we may employ these rules incorrectly. That is why there is retraction.

The application of that rule is retracted then. All consequences of that application are retracted as well. Given a premise set Γ one likes to know which of them may be abnormalities. *Abnormalities* here are, for instance, formulas of the form $\varphi \wedge \neg\varphi$. Some premise sets might be such that we know:

(1) $(A \wedge \neg A) \vee (B \wedge \neg B)$

whereas neither disjunct is a consequence (so far). So maybe each of them or either "A" or "B" behaves abnormally. The abnormalities form a set Δ. "Dab(Π)" abbreviates the disjunction of $(\varphi \wedge \neg\varphi)$ for all $\varphi \in \Delta$. "Dab(Π)" then expresses that *at least one* of the premises in Δ is abnormal. "Dab" means "disjunction of abnormalities". We are looking for *minimal* Dab-formulas (since the less disjuncts a Dab-formula has the more premises we have excluded as suspects). *Besides* the formulas appearing in a Dab-formula there might be formulas which are *already* known as being abnormal.

In general: $\Gamma \vdash_{\mathrm{LLL}} (\varphi \vee \mathrm{Dab}(\Pi))$ iff $\Gamma \vdash_{\mathrm{ULL}} \varphi$

Here Π contains the formulas on the consistency of which the application of some rules used in deriving φ depends. One can follow the *Minimal Abnormality* strategy, which with respect to (1), for example, assumes that once we consider the one abnormal we can take the other as normal (i.e., we can derive more consequences, since less exceptions are now operative). If at some later stage in a proof one can derive one of the disjuncts in (1) [in general in: Dab(Π)], then (1) [or Dab(Π)] is no longer *minimal*. So this Dab-formula is replaced by one stating that derived inconsistency. Retractions based on the supposed inconsistency of one of the other disjuncts are taken back then (by *marking/unmarking* lines in the proof, see below).

Proofs look like Natural Deduction Proofs with a further column:

n.<k,...> A Rule, m, l {B}

We number the lines and include in "< >" the premises a line depends on, then follows the formula, then a column naming the rule applied to get this line and the lines used in that application. The fifth column contains the set of formulas (possibly empty) on the consistency of which the derivability of the formula depends. These sets are called "conditions" (or "presuppositions", see below).

Conditions obey the following abstract rules:

(RU) If $\varphi_1 \ldots \varphi_n \vdash_{LLL} \psi$, then from $\varphi_1 \ldots \varphi_n$ on the conditions $\Pi_1 \ldots \Pi_n$ derive ψ on the condition $\Pi_1 \cup \ldots \cup \Pi_n$.

The rule (RU) concerns rules of Natural Deduction which do not require in LLL the consistency of the ingredient formulas. ψ just inherits the conditional dependencies.

Rules requiring such consistency operate on

(RC) If $\varphi_1 \ldots \varphi_n \vdash_{LLL}(\psi \vee Dab(\Pi_m))$, then from $\varphi_1 \ldots \varphi_n$ on the conditions $\Pi_1 \ldots \Pi_n$ derive ψ on the condition $\Pi_m \cup \Pi_1 \cup \ldots \Pi_n$

In this case consistency assumptions for the formulas in Π_m are *added*. The last line of a proof is the stage that the proof has arrived at. Now, if one of the formulas in the condition gets to be known as non fulfilling the essential criterion (here: consistency) the line is marked. The marking rule of the Minimal Abnormality strategy says roughly: If for $\varphi \in \Pi_i$, φ occurs in some Dab-formula, then line *i* is not marked because of that Dab-formula if there is another disjunct of that Dab-formula which is taken as unreliable. Lines that depend on a marked line have inherited the condition by either (RU) or (RC) and are, therefore, marked as well. Depending on the strategy – or the premise set – a line can get unmarked later, even in case of the reliabilist strategy.

"< >" notes the assumptions a line depends on. We note the result of a (vertical) derivation in a (horizontal) formula by putting the assumptions mentioned in the dependency set of the last derived line on the left of "⊢".

1.<1>	A	PREM
2.<2>	A ⊃ B	PREM
3.<1.2>	B	(⊃E) 1,2

that is A, A⊃B ⊢ B.

Instead of simply writing "(RU)", "(RC)" the detailed rules are given here.

1.<1>	¬ A ∧ C	PREM	∅	
2.<2>	B ⊃ A	PREM	∅	
3.<3>	D ∨ ¬ C	PREM	∅	
4.<4>	C ⊃ A	PREM	∅	
5.<5>	A ∨ ¬ C	PREM	∅	
6.<1>	¬ A	∧E, 1	∅	(RU)
7.<1>	C	∧E, 1	∅	(RU)
8.<1,2>	¬ B	Contraposition, 6, 2	{A}	(RC) marked at 10
9.<1,3>	D	∨E, 3, 7	{C}	(RC) if marked at 10 unmarked at 11
10.<1,5>	(¬ A ∧ A) ∨ (¬ C ∧ C)	Dilemma, ∧I, 5, 6, 7	∅	(RU)
11.<1,4>	¬ A ∧ A	⊃E, 4, 7	{C}	(RC)

In line 10 we get to know that at least one of "C" and "A" is inconsistent, so lines depending on them get marked. Given a Minimal Abnormality strategy *or* seeing in line 11 that "A" is inconsistent we can blame "A" for line 10 and unmark the lines depending on the consistency of "C". The Dab-formula in 10 is no longer *minimal* after 11.

Given the dynamic character of the proofs one has to distinguish: derivability at some stage and final derivability. φ is *finally derived* at line *i* of a proof at a stage *s* iff line *i* is unmarked at *s*, and whenever line *i* is marked in an extension of the proof, then there is a further extension in which line *i* is not marked. This property is (in most cases) not recursive. Even if final derivability is not recursive this resembles our actual reasoning where we (mostly) lack similar assurance against revision. There is *nothing* dynamic about *final* derivability. The *relative derivability statements* (i.e. those statements like

⊢((¬A∧C)∧(B⊃A)∧(D∨¬C)∧(A∨¬C)) ⊃ D   *given* {°C}

expressing that something is derivable from a (empty) set of premises on the given set of presuppositions) *are* recursive enumerable. So one should not exaggerate the failure of enumerability of theorems!

We built the system for adaptive paraconsistent set theory by using the following ingredients:

- an adaptive version of standard propositional calculus;
- semantic and consistency operators
- quantifier rules in the vain of Minimal Free Description Theory (allowing for the use of descriptions, including those that are non-referring);
- identity rules that restrict substitution to consistent objects;
- rules for a stronger conditional, whether we really need this or not;
- basic rules for modalities;
- set theoretic rules/axioms.

Semantic and consistency operators express within the language some of the semantic properties of sentences of the language.

A	¬A	TA	FA	ΔA	∇A	∘A	•A
0	1	0	1	0	1	1	0
1	0	1	0	1	0	1	0
0,1	0,1	1	1	0	0	0	1

These operators express: true, false, true only, false only, non-contradictory, contradictory.

We call the system resulting from these logical rules with added set theoretical rules **APS** (Adaptive Paraconsistent Set Theory).

A line that reads

n.< >   A

contains a theorem, since the sentence “A” does not depend on any assumption (the dependency set noted within “< >” is empty).

Theorems can be introduced into derivations at any time.

[Letters “n”, “m” etc. are used to refer to unspecified line numbers. Remember: “A” is an abbreviation, the object language having really sentences like “x∈y”, “Ordinal(ω)” etc.]

To include **PC**-tautologies, which we know already, we have the rule:

(PC) n.< > A   PC   ∅

where “A” is any PC-theorem. The column with markings is empty.

For any other theorems (i.e. already proven **APS**-theorems) we have:

(TH) n.< > A TH Γ

where "A" is any **APS**-theorem. Γ contains the presuppositions. There cannot be a list of marked individual constants in theorems.

To introduce assumptions into a derivation we have the rule:

(AE) n.<n> A AE {sat(A)}

where we define satisfiability presuppositions by the schema

(sat) $\mathrm{sat}(A) \equiv \neg((\Delta A \wedge FA) \vee (\nabla A \wedge TA) \vee (\bullet A \wedge {}^{\circ}A))$

In case the presupposition later turns out to be violated lines depending on the assumption in question have to be retracted (as always). The satisfiability presupposition has to be made because the definition of **APS**-consequences excludes the cases in which the premise set is unsatisfiable [see below]. Typically assumption in arguments need not be considered really true, but satisfiable at least.

Conjunction Introduction has the form:

n.<m>	A	...	Γ
o.<k>	B	...	Λ
p.<m,k>	$A \wedge B$	($\wedge$I) n, o	$\Gamma \cup \Lambda$

Conjunction Elimination has the two forms[42]:

n.<m>	$A \wedge B$	...	Γ
o.<m>	A	($\wedge$E) n	Γ
n.<m>	$A \wedge B$	...	Γ
o.<m>	B	($\wedge$E) n	Γ

Disjunction Introduction has the two forms:

n.<m>	A	...	Γ
o.<m>	$A \vee B$	($\vee$I) n	Γ
n.<m>	A	...	Γ
o.<m>	$B \vee A$	($\vee$I) n	Γ

[42] Here and in the following rules "<m>" refers to an unspecific (number) of assumptions that the line depends on. Γ can, of course, be empty; if there are marked individual constants they are marked only in the line where the quantificational rule is employed, see below.

Disjunction Elimination has the form:

n.<m>	$A \vee B$	...	Γ
o.<k>	$\neg A$	...	Λ
p.<m,k>	B	($\vee$E) n,o	$\Gamma \cup \Lambda \cup \{°A\}$

This is the restricted form of Disjunctive Syllogism.

Negation Introduction has the form:

n.<n>	A	AE	$\varnothing$
o.<m,n>	$\neg A$	...	Γ
p.<m>	$\neg A$	($\neg$I) n,o	Γ

If some assumption allows deriving its own negation, then this sentence can be stated negated *simpliciter* (i.e. the status as assumption is discharged, as indicated by the underlining in the line using ($\neg$I)). The usual form of Negation Introduction leads to trivialization in inconsistent contexts, and thus cannot be adopted here.

Negation Elimination has the form:

n.<m>	$\neg\neg A$	...	Γ
o.<m>	A	($\neg$E) n	Γ

Conditional Introduction (Conditionalization) has the form:

n.<n>	A	AE	$\varnothing$
o.<m,n>	B	...	Γ
p.<m>	$A \supset B$	($\supset$I) n,o	Γ

This rule mirrors the Deduction Theorem. If the conditionalization is the last step of a derivation the restrictions on not having marked individual constants in it have to be kept.

Conditional Elimination (*Modus Ponens*) has the form:

n.<m>	$A \supset B$	...	Γ
o.<k>	A	...	Λ
p.<m,k>	B	($\supset$E) n,o	$\Gamma \cup \Lambda \cup \{°A\}$

This is the restricted form of Modus Ponens.

Truth Introduction/Elimination follow the disquotational (T)-schema.[43] Strict Falsity will be a defined notion. The Inconsistency operator is treated by rules as well.

[43] One may doubt that the operator "T" thus can correspond to "is true" in a substantial and especially in an (mildly) epistemic conception of truth, which does not

Truth Introduction has the form:

n.<m>	A	...	Γ
o.<m>	TA	(TI) n	Γ

Truth Elimination is the converse:

n.<m>	TA	...	Γ
o.<m>	A	(TE) n	Γ

Falsity Introduction has the form:

n.<m>	$\neg A$	...	Γ
o.<m>	FA	(FI) n	Γ

Falsity Elimination is the converse:

n.<m>	FA	...	Γ
o.<m>	$\neg A$	(FE) n	Γ

Inconsistency Introduction has the form:

n.<m>	$A \wedge \neg A$	...	Γ
o.<m>	$^{\bullet}A$	($^{\bullet}$I) n	Γ

Inconsistency Elimination is the converse:

n.<m>	$^{\bullet}A$	...	Γ
o.<m>	$A \wedge \neg A \wedge \neg{}^{\circ}A$	($^{\bullet}$E) n	Γ

For strict truth we introduce its version of *Convention (T)*:

n.<>	$\Delta A \equiv A$	(Δ)	$\varnothing$

Necessity Introduction (Necessitation) has the form:

n.< >	A	...	Γ
o.< >	$\Box A$	($\Box$I) n	Γ

A theorem (but not any sentence depending on further assumptions) can be necessitated.

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validate $\varphi \supset T\varphi$. With respect to the evaluation of formula there is, however, this operator, and it serves sometimes the function of "is true".

Necessity Elimination has the form:

n.<m>	□A	...	Γ
o.<m>	A	...	Γ

Since necessity is taken here to be semantic necessity (not natural necessity or some more restricted version of necessity) it has to be governed in the way of a normal modal logic of the strength of modal system **S5**. Therefore we need two further rules:

The rule corresponding to the **K**-Axiom of modal logic has the form:

n.<m>	□(A ⊃ B)	...	Γ
o.<k >	□A ⊃ □B	(K) n,o	Γ

The rule corresponding to the **S5**-Axiom has the form:

n.<m>	◇A	...	Γ
o.<m>	□◇A	(S5) n	Γ

Taking entailment to be semantic entailment in the sense that:

A ⥽ B ≝ □(A ⊃ B)

gives us derived introduction and elimination rules for "⥽".

Entailment Introduction is a strict form of Conditionalization:

n.<n>	A	AE	∅
o.<o>	B	AE	∅
…			
r.<n,o>	C	...	Γ
s.< >	A ∧ B ⥽ C	(⥽I) n,o,r	Γ

In strict conditionalization all assumptions have to be conditionalized (thus we get a theorem to be necessitated to yield the entailment).

Entailment Elimination is a version of *Modus Ponens*:

n.<m>	A ⥽ C	...	Γ
o.<k>	A	...	Λ
p.<m,k>	C	(→E) n,o	Γ ∪ Λ ∪ {°A}

We introduce some further connectives by definitions. There are derivable introduction and elimination rules then. Within a derivation we use the definitions by referring to their name:

(D≡)	A ≡ B ≝ (A ⊃ B) ∧ (B ⊃ A)
(D⥼⥽)	A ⥼⥽ B ≝ (A ⥽ B) ∧ (B ⥽ A)
(D◇)	◇A ≝ ¬□¬A
(D∇)	∇A ≝ Δ¬A

(D°) $°A := \Delta A \vee \nabla A$

We have to give the usual requirements on marking individual terms in case of applying Universal Generalization or Existential Specialization within a derivation. These are:

- Terms generalized in Universal Generalization and specialized to in Existential Specialization are marked at the right of such a line;
- The marking also notes the dependencies on other individual terms in that line (in the form "a(e)": "a" being marked depended on "e");
- Markings may not be circular (i.e. we do not have "a(e)" and "e(a)");
- No term may be marked twice;
- Marked terms may neither occur in the premises, presuppositions nor in the conclusion of a supposed valid derivation.

In applications of the quantifier rules one also has to meet the requirement that by generalising one constant to a variable "x", "x" will not be bound by already present quantifiers. (∀I) and (∃I) require further on that "x" and the individual term occur at exactly the same places in a given sentence.

E!(á) says that the object denoted by á exists, "E!( )", being the *existence predicate*, is a logical constant. Quantifiers refer to existing objects only.[44] We assume that there is something:

<u>Axiom of Existence</u>

n.< > (∃x)E!(x) (E!) ∅

<u>Identity Introduction</u> is valid for any object, existing or not:

n.< > a = a (=I) ∅

Identity Elimination (i.e. substitution of identicals) is more critical. It has to be restricted to avoid trivilization in a paraconsistent logic with as much

[44] Semantically speaking the extension of "E!( )" is the domain at a world index. As we are dealing with sets anyway worlds may be taken as sets containing set theoretically modelled facts or states of affairs. Whether one admits *possibilia* or not is a question to be discussed apart from set theoretical assumptions. [There are several techniques to avoid a commitment to *possibilia* in one's semantics. For the non-modal case one may take the interpretation function *I* on terms to be partial: If *I* is defined for α, ||α|| is in the domain, "E!(α)" is true, if *I* is not defined for α. "E!(α)" is false; *I* interprets P(á) for any general term and any singular as true, false or both; complex statements have their usual recursive truth conditions (like in **LP**); variable assignments run over the domain, thus providing the usual (paraconsistent) quantificational semantics, even if there are objects without names; for any term α "α = α" is true, if *I* is defined on both α and γ, the usual truth condition for "=" applies, otherwise a value may be assigned at random. In the modal case modal expressions have their usual truth conditions (like in **S5**), *possibilia* can then only be avoided by some construction of an 'outer domain' of terms and some 'ersatzist' construction involving instantiating terms.]

expressive power as **APS**. We have to presuppose that some object is not an inconsistent object to apply (=E) to it. We define a consistency predicate "K( )" for objects (as a logical constant, of course) to do this:

(DK) $K(á) \stackrel{def}{=} \neg(\exists P)(P(á) \wedge \neg P(á))$

APS is no 2nd order system, but we may employ (DK) in that we note K(á) in the presupposition list of some line if for the object named á we should not have a line with an instance of the scheme: $P(á) \wedge \neg P(á)$. A line with K(á) presupposed will be retracted once we derive $P(á) \wedge \neg P(á)$ for some predicate.

<u>Identity Elimination</u> then takes the form:

n.<m>	P(á)	...	Γ
o.<k>	á = é	...	Λ
p.<m,k>	P(é)	(=E) n,o	$\Gamma \cup \Lambda \cup \{K(é)\}$

This restriction may block deriving theorems concerning inconsistent sets, like the restriction on ($\supset$E) blocks theorems concerning contradictions. Should we bother? We need not believe that inconsistent sets are like consistent sets. After all the point of **APS** may be seen to rest in dealing with lurking inconsistent sets in an attempt to have a most naïve set theory for consistent sets (i.e. one with unrestricted Comprehension and U). It is not obvious that **ZFC**-like axioms should apply to inconsistent sets. It is not obvious - in fact it may be doubted – that our concept set applies in full generality to inconsistent sets. **APS** *can* handle inconsistent sets if there are any. If it turned out that none can be *shown* to exist (by restrictions on proofs like restricting detachment in Naïve Comprehension to consistent set defining formula) so the better, we may presume [see below on the Russell Set]. The priority with **APS** lays on realizing a naïve set theory for consistent sets.

Since we want to use description and modal operators we have to provide (=E) with a provisio in case descriptions are involved. In modal logic **S5** all modalities can be reduced to modalities of degree 1. We require as a *provisio* for Identity Elimination:

In case we have á = é, then:

if á is a description and é an individual constant, é cannot be substituted into a modal context of "◇",

if á is an individual constant and é a description, é cannot be substituted into a modal context of "□".

The following quantifier rules require following the rules of marking the constant generalized/specialized in ($\forall$I) and ($\exists$E), and the renaming of variables mentioned before.

<u>∀-Introduction</u> (Universal Generalization) has the form:

n.<m>	R(á,é)	...	Γ	
o.<m>	(∀x)R(x,é)	(∀I),n	Γ ∪ {E!(á)}	á(é)

Thus the application of (∀I) requires an existence assumption concerning á, since we conclude to a generalization about all existing objects. á is marked, here as depending on é.

<u>∀-Elimination</u> (Universal Instantiation) has the form:

n.<m>	(∀x)P(x)	...	Γ
o.<m>	P(é)	(∀E),n	Γ ∪ {E!(é)}

Since the generalization is (maybe) true of existing objects only the application of (∀E) presupposes that the constant specialized to names an existing object.

<u>∃-Introduction</u> (Existential Generalization) has the form:

n.<m>	P(á)	...	Γ
o.<m>	(∃x)P(x)	(∃I),n	Γ ∪ {E!(á)}

Thus the application of (∃I) requires an existence assumption concerning á, since we conclude to a generalization about some existing objects.

<u>∃-Elimination</u> (Existential Instantiation) has the form:

n.<m>	(∃x)R(x,á)	...	Γ	
o.<m>	R(é,á)	(∃E),n	Γ ∪ {E!(é)}	é(á)

Since the generalization is (maybe) true of existing objects only the application of (∃E) presupposes that the constant specialized to names an existing object. The name of the object is marked in its dependencies in the formula in question.

In case that existence assumptions are explicitly made the existence presupposition can be *cancelled*:

n.<m>	P(á)	...	Γ ∪ {E!(á)}
o.<o>	E!(á)	AE	
q.<m,o>	P(á)	(E!C),n,o	Γ

If the existence claim follows from the other assumptions the presupposition can be cancelled as well:

n.<m>	P(á)	...	Γ ∪ {E!(á)}
o.<m>	E!(á)	…	Γ ∪ {E!(á)}
q.<m>	P(á)	(E!C),n,o	Γ

Minimal Free Description Theory requires the uniqueness of a description with respect to the existing objects only. Otherwise it looks like the standard Russellian account of descriptions. We use the usual “ι”-notation, so that “ιxF(x)” means “the (unique) F”.

<u>The (MFD)-rule</u> can be stated as the following two ways of term interchangability:

n.<m>	ιxP(x) = á	...	Γ
o.<m>	(∀y)(á=y ≡ P(y) ∧ (∀z)(P(z) ⊃ z=y)	(MFD),n	Γ
n.<m>	(∀y)(á=y ≡ P(y) ∧ (∀z)(P(z) ⊃ z=y)	...	Γ
o.<m>	ιxP(x) = á	(MFD),n	Γ

The first conjunct in the equivalence states satisfaction of the defining property, the second expresses uniqueness.

In the context of quantificational rules we can now make clear the reference to a set of presuppositions above. Adaptive Logics speak of *Dab*-formula and corresponding sets of consistency assumptions. **APS** notes these consistency assumptions as presuppositions to employ some restricted rules. Actually the consistency presupposition is "°A". In Minimal Free Description Theory usually a conjunct "E!(a)" is needed (e.g. as derivable line or assumption) to employ one of the quantifier rules. Since **APS** is a dynamic logic already we need not work with "E!(a)" as a line in a derivation, but can note this also as a presupposition in the presupposition set Γ noted on the right. In case of Identity Elimination the presupposition is that we have a consistent object. We note this as the presupposition “K(a)” for an object *a* in question. Each of the sentences in the presupposition set has a negation. Once the negation of such a presupposition can be derived, all lines are retracted which depend on that presupposition (like in the original adaptive dynamics). The retraction thus does not only concern the disappointment of consistency assumptions (either for a sentence or an object), but also the disappointment of existence presuppositions. If the last line φ of a derivation has a non-empty presupposition set Γ, this means that the sentence in that line is derivable from the assumptions noted within “< >” given these further presuppositions.

Let Φ be the (possibly empty) set of assumptions and Γ the (possibly empty) set of presuppositions in a derivation of φ.

We have:

$$\neg(\exists\psi\in\Gamma)\ \Phi\vdash_{APS} \neg\psi \Rightarrow \Phi\vdash_{APS} \varphi$$

To save labour and have derivation looking more closely like standard derivations we adopt the convention to drop noting Γ if Γ is empty.[45]

The *relative derivability statements,* i.e. those statements like

$\vdash_{APS} G(\iota xF(x)) \supset (a = \iota xF(x) \supset G(a))$ *given* {K(a)}

expressing that something is derivable from a (empty) set of premises on the given set of presuppositions, *are* recursive enumerable. Noting presuppositions explicitly clutters derivability statements, one may complain. This is due, however, on the universal employability of **APS**. Standard logics have all these caveats implicitly understood as they presuppose a well-behaved restricted area of applications.

Consequence in **APS** may be defined:

($\vDash_1$) $\Gamma \vDash_{APS} \varphi$ iff
in case that *all* $\psi \in \Gamma$ are true at least, then φ is *true at least*.

Nothing needs to be said concerning the case that any $\psi \in \Gamma$ is false *only*. One has not to hold that then a consequence relationship holds. To do so would endorse non-relevant inferences.

To do so may come close to reintroducing *ex contradictione quodlibet*, as well. ΔA and ∇A are incompatible, so both can never be true at the same time, so allowing for Irrelevant consequences would yield, for example:

(*2) $\nabla A, TA \vDash_{APS} C$

for any C.

To insist that the "in case" has to be read as material implication as in **PC** just begs the questions against a relevant meta-theory!

An improved relevant definition of consequence in **APS** might be:

($\vDash_2$) $\Gamma \vDash_{APS} \varphi$ iff *there are* models such that all $\psi \in \Gamma$ are true at least, and in case that all $\psi \in \Gamma$ are true at least in a model, then φ is true at least in that model.

The existence condition rules out the Irrelevant cases and (*2). A consequence relation obtains if and only if all of the non-empty set of models that make the premises at least true make the consequence at least true. What models are has to be explained in our universal system **APS** itself.
As always ⊨ concerns the inheritance of truth. The second version, ($\vDash_2$), requires some reworking of the proof theory.

[45] It may also be convenient to leave most presuppositions, especially satisfiability presuppositions, as being understood and return to them only in case of cancellation of lines. For the sake of getting to see all the presuppositions or to accustom to the adaptive procedures it may be useful to write them down for a while.

Changing the definition of consequence this way requires a further book keeping of presuppositions, in this case with respect to assumptions.

In as much as **APS** has to be *correct* the basic rules must not support consequence claims that go against the definition above. Making an unsatisfiable assumption, however, would allow claims like

(*3) $\nabla A \wedge TA \vDash_{APS} \nabla A \wedge TA$

(*4) $\nabla A \wedge TA \vDash_{APS} \nabla A$

where the premise (set) is unsatisfiable and thus the claims are *supposedly* incorrect. If we consider these claims as *incorrect* – and not just non-relevant – then the first definition of consequence is in trouble, since conjunction elimination would allow to derive (*4). The proof theory allows to derive something that is not – strictly speaking – a violation of the definition of consequence given thus, but only because we deem it non-relevant (the case of the assumption on the left being at least true just does not arise, thus it cannot violate the condition). This line of reasoning, however, leads to accepting (*2) as *not incorrect*! And this may be too much, even if (*2) is not accepted as *valid*. Still the first definition may be an option given a clear understanding of RELEVANCE. Being silent on (*2) as neither correct not incorrect, however, violates the otherwise assumed *tertium non datur*, and the meta-theory should not work with another logic than the logic, since a truly universal logic can be used as its own meta-logic.

Clearly, however, the solution for this first option's trouble is straight forward, given the second definition: assumptions (i.e. claims to be considered for further consequences) are presupposed not to be true, but to be *satisfiable*. In a paraconsistent semantics even contradictions $\varphi \wedge \neg\varphi$ can be satisfiable.

When applying the **assumption rule** (**AE**) we have to use the form

n.<n> A AE sat({A})

where we define the satisfiability presuppositions by the schema

(sat) sat(Γ) ≡ Γ has a **APS**-model where all $\varphi \in \Gamma$ are true at least

Γ being a set of assumptions. The set of assumptions Γ has to be *jointly satisfiable*. With sat({A}) we note only the satisfiability of an *individual* assumption. If a line depends on several assumptions, the further assumptions entering into its derivation also have to enter the set the satisfiability of which is presupposed. The presupposition of satisfiability is cancelled when Γ contains or entails for some A either

(i) $TA \wedge \nabla A$ or

(ii) $\nabla A \wedge \Delta A$ or

(iii) $\Delta A \wedge FA$ or

(iv) $(\bullet A \wedge {}^{\circ}A)$

These sentences are beyond contradictions like $\varphi\wedge\neg\varphi$ in not being satisfiable even in a paraconsistent semantics for **APS**.

In case the presupposition later turns out to be violated lines depending on the assumption in question have to be retracted (as always). In a universal logic like **APS** were the distinction between object and meta-language is superseded by the idea of semantic closure we naturally have *semantic* properties (like satisfiability) enter into the *syntactic* properties of a derivation.

Since we *generally* have to presuppose the satisfiability of the set of assumptions which a line depends on, we may use the convention *of not especially noting this* in ordinary cases, but proceed according to a revision rule that all lines depending on an assumption that turned out to be unsatisfiable have to be taken back. In fact the additional entry "sat(A)" for some premise "A" is *redundant* in our derivations as we note the dependencies *in the second column*. We just have to recognize that all premises mentioned in the second column have to be satisfiable. This is different with the other presuppositions, as, for instance, not *all* premises have to be consistent.

The relative derivability statements with respect to logical consequence (i.e. derivability from a set of assumptions) now carry the *presupposition* that the assumptions/premises are satisfiable (in the defined sense above):

$A \wedge B \vdash_{APS} A$ given $\text{sat}\{(A \wedge B)\}$

expressing that something is derivable from a satisfiable set of premises. So in the next few examples the "sat" is used, but it can be dropped for more convenient representation. More generally one could say that in a claim like $\varphi\wedge\psi\vdash_{APS}\varphi$ it is meant that in case of $\varphi\wedge\psi$ being satisfiable φ is derivable. Here $\varphi\wedge\psi\vdash_{APS}\varphi$ is a general schema, but one need not be committed to every instance of $\varphi\wedge\psi$ providing a true statement of derivability $\varphi\wedge\psi\vdash_{APS}\varphi$ as in case of (*4).

<u>Examples</u> of **APS**-derivations:

1.$\langle 1\rangle$	$G(\iota xF(x))$	AE	$\{\text{sat}(G(\iota xF(x)))\}$
2.$\langle 2\rangle$	$a = \iota xF(x)$	AE	$\{\text{sat}(a = \iota xF(x))\}$
3.$\langle 1,2\rangle$	$G(a)$	(=E)	$\{K(a),\text{sat}(1),\text{sat}(2)\}$
4.$\langle 1\rangle$	$a = \iota xF(x) \supset G(a)$	$(\supset I)\underline{2},3$	$\{K(a),\text{sat}(1)\}$
5.$\langle\rangle$	$G(\iota xF(x)) \supset (a = \iota xF(x) \supset G(a))$	$(\supset I)\underline{1},4$	$\{K(a)\}$

1.<1>	$a = \iota x F(x)$	AE	$\{sat(1)\}$
2.<1>	$(\forall y)(a=y \equiv F(y) \wedge (\forall z)(F(z) \supset z=y)$	MFD,1	$\{sat(1)\}$
3.<1>	$a=a \equiv F(a) \wedge (\forall z)(F(z) \supset z=a)$	$(\forall E)2$	$\{E!(a),sat(1)\}$
4.<>	$a=a$	(=I)	$\varnothing$
5.<1>	$F(a) \wedge (\forall z)(F(z) \supset z=a)$	$(\supset E)3,4$	$\{E!(a),{}^\circ(a=a),sat(1)\}$
6.<1>	$F(a)$	$(\wedge E)5$	$\{E!(a),{}^\circ(a=a),sat(1)\}$
7.<1>	$(\exists x)F(x)$	$(\exists E)6$	$\{E!(a),{}^\circ(a=a),sat(1)\}$
8.<>	$a = \iota x F(x) \supset (\exists x)F(x)$	$(\supset E)\underline{1},7$	$\{E!(a),{}^\circ(a=a)\}$

1.<1>	A	AE	$\{sat(1)\}$
2.<>	$A \supset \neg\neg A$	PC	$\varnothing$
3.<1>	$\neg\neg A$	$(\supset E)1,2$	$\{{}^\circ A,sat(1)\}$
4.<4>	$\neg A \vee B$	AE	$\{sat(4)\}$
5.<1,4>	B	$(\vee E)3,4$	$\{{}^\circ A,{}^\circ\neg\neg A,sat(1),sat(4)\}$
6.<4>	$A \supset B$	$(\supset I)\underline{1},5$	$\{{}^\circ A,{}^\circ\neg\neg A,sat(4)\}$
7.<>	$(\neg A \vee B) \supset (A \supset B)$	$(\supset I)\underline{2},6$	$\{{}^\circ A,{}^\circ\neg\neg A\}$

1.<>	$\Diamond\neg A \supset \Box\Diamond\neg A$	$(S5)(\supset I)$	$\varnothing$
2.<>	$\neg\Box\Diamond\neg A \supset \neg\Diamond\neg A$	$(PC)(\supset E)1$	$\{{}^\circ(\Diamond\neg A \supset \Box\Diamond\neg A)\}$
3.<>	$\Diamond\Box A \supset \Box A$	$(D\Diamond)2$	$\{{}^\circ(\Diamond\neg A \supset \Box\Diamond\neg A)\}$
4.<>	$\Box A \supset \Diamond\Box A$	(TH)	$\varnothing$
5.<>	$\Box A \supset \Box\Box A$	$(\supset E)(\supset I)3,4$	$\{{}^\circ\Box A,{}^\circ(\Diamond\neg A\supset\Box\Diamond\neg A)\}$
6.<6>	$\Box A$	AE	$\{sat(6)\}$
7.<6>	A	$(\Box E)6$	$\{sat(6)\}$
8.<6>	$A \vee B$	$(\vee I)7$	$\{sat(6)\}$
9.< >	$\Box A \supset A \vee B$	$(\supset I)\underline{6},8$	$\varnothing$
10.< >	$\Box(\Box A \supset A \vee B)$	$(\Box I)9$	$\varnothing$
11.<>	$\Box\Box A \supset \Box(A \vee B)$	(K)10	$\varnothing$
12.<>	$\Box A \supset \Box(A \vee B)$	$(\supset E)(\supset I)$ 5,11	$\{{}^\circ\Box A,{}^\circ(\Diamond\neg A\supset\Box\Diamond\neg A)\}$

We add to the *constants* of our language the expression “Set( )” with its obvious intended meaning, as well as the usual set theoretical symbols like curly brackets, “$\in$”, “$\subseteq$” etc. and “|” to express set abstracts like $\{x \mid P(x)\}$. Set abstracts are *terms* in the language.

We have to add the usual definitions like:

$$(D\subseteq 2)\qquad a \subseteq b \overset{\text{def}}{=} Set(a) \wedge Set(b) \wedge (\forall x)(x \in a \supset x \in b)$$

$$(D\langle\,\rangle)\qquad \langle a,b\rangle \overset{\text{def}}{=} \{\{a\},\{a,b\}\}$$

Thus “$\in$” is taken as primitive.

As **APS** deals not only with sets, but also ordinary objects, we need a set predicate "Set( )" to sort out the empty cases when using set principles with non-sets. Since in the intended interpretation x ∈ y will be false if *y* is not a set, we have to avoid all the counterintuitive consequences of irrelevant conditionals (e.g., (∀x,y)(x∈a ≡ y∈b) is true for any individuals *a* and *b* without these being identical).

We use an introduction rule for some of these cases:

n.<o>	á = {x ǀ P(x)}	…	Γ
m.<o>	Set(á)∧(∀x)(P(x) ≡ x∈á)	(Set),n	Γ

We define the *universal set* U, as we have done here all the time:

(DU) U $\stackrel{def}{=}$ {x ǀ x = x}

Since we cannot exclude inconsistent objects like *a* with *a* ≠ *a*, it is no option to define ∅ as {x ǀ x ≠ x}!

A better idea is:

(D∅) ∅ $\stackrel{def}{=}$ {x ǀ x ∉ U}

Thus U is truly universal.

Extensionality of sets can be added as a rule to introduce identity of sets.

n.<k>	Set(a) ∧ Set(b) ∧ (∀x,y)(x∈a ≡ y∈b)	…	Γ
m.<k>	a = b	(Ext),n	Γ

Substitution doing the rest for consistent sets, since both sides of "∈" are open for substitution.

Naïve Comprehension is added as rule/axiom *schema* using the material conditional:

n.< > (∃y)(Set(y) ∧ (∀x)(x∈y ≡ P(x))) (NC) ∅

with no further restrictions. (NC) immediately gives us the existence of ∅ and U. With (NC), (D⊆) and (Ext) we get:

- unordered pairs {x,y} by v∈w ≡ v=x ∨ v=y
- ordered pairs, then abbreviated as <x,y>
- singletons, {x} = {y ǀ y = x}

and so forth. Also by (SET) we immediately have: x∈{yǀ y = x} = {x}.

Given the restrictions on detachment in **APS**, however, may forbid or retract the application of detachment here. Consider, for example, the inconsistent object *a* with *a* ≠ *a*. We have

1.$\langle\ \rangle$ $a \in U \equiv a = a$	(NC), (DU), ($\forall$E)	$\varnothing$
2.$\langle\ \rangle$ $a \in U$	($\equiv$E), (=I), 1	$\{{}^\circ a=a\}$
3.$\langle 3\rangle$ $a \neq a$	AE	$\varnothing$
4.$\langle 3\rangle$ $a \neq a \wedge a = a$	($\wedge$E), (=I), 3	$\varnothing$

Now, given (4) "a = a" obviously isn't consistent, so (2) has to be retracted! That, of course, does not mean that $a \notin U$.

For the Russell Set we have by (NC): $(\exists y)(Set(y) \wedge (\forall x)(x \in y \equiv x \notin x))$; naming the set $\{x | x \notin x\}$ "R" we get by Set "Set(R)" and then by (NC) and ($\exists$E): $(\forall x)(x \in R \equiv x \notin x)$. So by ($\forall$E): $R \in R \equiv R \notin R$. We cannot get "$R \in R \wedge R \notin R$", however, as ($\supset$E) requires a consistent antecedent and "$R \in R$" turns out to be inconsistent. So we have introduced the set $\{x | x \notin x\}$ but have not derived the contradiction showing it to be inconsistent. This may be a case of incompleteness for **APS**: We cannot show all the properties of inconsistent sets. On the other hand – why should we bother? Only if we *assume* R to exist as inconsistent set, can we declare **APS** incomplete, supposing, of course, there being no other proof of "$R \in R \wedge R \notin R$". The issue here concerns only those interested in knowing the structure of inconsistent objects, as some dialetheist might be. They had to come up with a better system which does what **APS** does for consistent sets, but can also additionally treat more completely of inconsistent sets. Dealing with universality **APS** suffices.

We may allow shorthand expressions for functions: f, f'… As we can say:

(Df) $\quad Function(f) \stackrel{def}{=} Set(f) \wedge (\forall y)(y \in f \supset (\exists v,w)(y=\langle v,w\rangle)) \wedge (\forall y,y')(y \in f \wedge y' \in f \wedge (\exists v,w,w')(y=\langle v,w\rangle \wedge y'=\langle v,w'\rangle) \supset y=y')$

We define an injective function by:

(DInjective) $\quad Injective(f) \stackrel{def}{=} (\exists x,y)(Set(x) \wedge Set(y) \wedge (\forall u,u',v)(u \in x \wedge u' \in x \wedge v \in y \wedge \langle u,v\rangle \in f \wedge \langle u',v\rangle \in f \supset u = u')$

Powerset is defined in the usual way by:

(D$\wp$) $\quad \wp(a) \stackrel{def}{=} \{x | x \subseteq a\}$

And by (NC) we get for some set *w*:

$(\exists y)(Set(y) \wedge (\forall x)(x \in y \equiv x \subseteq w)$

Generalizing on *w* provides the Powerset Axiom.

We can define cardinality comparison $|a| \leq |b|$ now by

(D$\leq$) $\quad |a| \leq |b| \stackrel{def}{=} (\exists f)(\forall x \in a)(\exists y \in b)(f(x)=b \wedge Injective(f))$

"$>$" for cardinalities has then the obvious definition:

(D>) $|a| > |b| \overset{def}{=} \neg|a| \leq |b|$

We reason now:

1.< > $\wp(U) = \{x \mid x \subseteq U\}$ (D$\wp$),(NC)

2.< > $Set(\wp(U)) \wedge \wp(U) \subseteq U$ (Set),(DU),(D$\subseteq$), 1

3.< > $Injective(\{\langle x,y\rangle \mid x = y\})$ (DInjective),(Ext),(NC)

4.< > $(\exists f)(\forall x \in \wp(U))(\exists y \in U)(f(x)=y \wedge Injective(f))$ ($\exists$I),(D$\subseteq$),2,3

5.< > $|\wp(U)| \leq |U|$ (D$\leq$),4

6.< > $(\exists x)(Set(x) \wedge |\wp(x)| \leq |x|)$ ($\exists$I),(Set),5

7.< > $(\exists x)(Set(x) \wedge \neg|\wp(x)| > |x|)$ (D>),6

8.< > $\neg(\forall x)(Set(x) \supset |\wp(x)| > |x|)$ ($\forall\exists$), 7

where the last line, (TP1), is the negation of *Cantor's Theorem*. The usual indirect proof of *Cantor's Theorem* does not work in **APS** (like in many if not all paraconsistent logics). If there was another proof *Cantor's Theorem* would come out as an antinomy!

We can continue and observe: for a set *x* there exists by (NC) $f{:}x \rightarrow \wp(x)$ defined as $\{z \mid (\exists y \in x) z = \langle y,\{y\}\rangle\}$, the *singleton map* of the set *x*. By reasoning like the proof just considered we can arrive at:

10.< > $|U| \leq |\wp(U)|$

(5) and (10) combined with the *Cantor-Bernstein Theorem*

(CBT) $|a| \leq |b| \wedge |b| \leq |a| \supset |a| = |b|$

prove

11.< > $|U| = |\wp(U)|$ ($\wedge$I),($\supset$E),5,10,(CBT) $\{°((5)\wedge(10))\}$

which contrasts nicely with **NF/U**.[46]

Because the singleton map exists, we can prove:

(TP2) $(\forall x)(|x| \leq |\wp_1(x)|)$

And since $\varnothing \in \wp(x)$ for any set *x*, $\wp_1(x) \subset \wp(x)$ for any set *x*, so for *finite* sets we have:

(TP3) $(\forall x)(|x| < \aleph_0 \supset |x| < |\wp(x)|)$

Without *Cantor's Theorem* to generate higher cardinalities the **APS**-universe might be rather flat than V-shaped.

APS contains non-referring singular terms, using a name does not imply that the named object does exist. Singular terms naming sets – including set

[46] The *Cantor-Bernstein Theorem* can be proven directly, so the standard proofs are available in **APS**.

abstracts! – thus need not refer by just being singular terms. In principle there would therefore be the option for *non-existent sets* being around. These may one remind of Quine's 'virtual sets' [cf. Chap. III]. Virtual sets, with Quine, are set abstracts which are not quantified over, thus not being said to exist. **APS** could provide a place for such virtualities.

On the other hand (NC) just declares that any set whatsoever (i.e. any set defined by a set abstract) exists.

One may consider whether the introduction of a second set of quantifiers (quantifying over *possibilia* or *virtualities* as well) might be useful, using a quantifier with no existential impact in (NC). Apart from the problems of an ontology of *virtualities* this seems, to me, to be against the spirit of Naïve Comprehension, the very point of which seems to be that there (really) *is* a set to each defining condition.

Unrestricted (NC) gives us

$$(\exists y)(\forall x)(x \in y \equiv x = y)$$

i.e. a set $y = \{y\}$. *y* is its own singleton, thus finite. We have:

(T) There are finite self-membered sets.

This supposedly obvious observation is interesting as is has been conjectured for **NF** that any self-membered set in **NF** has to be infinite.

All the ordinals are members of

$$\Omega = \{x \mid x \text{ is the order-type of a well-ordered set}\}$$

Where, as usual, $\{\langle x,R\rangle \mid \langle x,R\rangle \text{ is isomorphic to } \langle y,R'\rangle\} = o$ is the order-type of $\langle y,R'\rangle$ with $\langle y,R'\rangle$ being the set *y* with *R*' well-ordering *y*. Ω is not just self-membered, but contains all its own ordinal successors! Ω contains all well-orderings, and for all (infinite, pure) sets there is a well-ordering in Ω as (NC) provides for each (infinite, pure) set a choice function, which can be employed to order the set. Finite sets can be well-ordered any way by counting. Uncountable infinite sets with urelements (i.e. non-sets) may be well-ordered by first ordering the finitely many urelements and then well-ordering the rest of the set.

All in **APS** seems to be a most comprehensive system for paraconsistent reasoning including reasoning about sets, recapturing standard theorems for consistent contexts and entities.[47]

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[47] (Dunn 1988) showed a somewhat disturbing result for non-classical logic and thus for non-classical set theory: If one combines a couple of basic and innocent principles for the classical connectives and for the consequence relation (like transitivity) with second order quantification and conversion principles (for a λ-calculus like abstraction) then the resulting logic (i.e. the set of theorems generated) is an extension of (standard) **SOL**. This means that the full strength – and paradox yielding power – of classical reasoning (including Disjunctive Syllogism …) is regained. The systems by Routley

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Paraconsistent set theory is not only of interest in itself or as formalization of naive set theory. It also may serve as the foundation of paraconsistent arithmetic. It has to be checked what remains of standard mathematics once its foundations in set theory have been restricted to paraconsistent set theories!

Do we need paraconsistency for set theory?

If the justification for a paraconsistent set theory depends on avoiding the antinomies, others approaches that avoid the antinomies might be alternatives. Provided they are more natural or coherent that the paraconsistent systems.

If the justification depends on the argument that the notion of set is not clear unless we have a universal set, set theories that combine standard logic with having a universal set might be alternatives. Again, provided they are more natural or coherent that the paraconsistent systems.

An even more decisive point may be keeping unrestricted (Naive) Comprehension. That is something that neither **ZFC** nor systems like **NF** can do. Paraconsistent set theories help Logicism to a second chance. Russell and others worried – *inter alia* – that the Axiom of Infinity does not sound like a logical principle, but boldly asserts the existence of specific sets. One we have (NC) we get U and ∅, and we can comprehend specific sets as subsets of U. There are infinite sets (in Dedekind's classical definition) as U can be mapped to the singleton image of its elements, thus is (Dedekind) infinite.

Neither does everyone like the distinction between classes and sets. A theory not making this difference might be preferred.

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and Brady outlined in the preceding paragraphs and versions of set theory framed in first order **LP** or **APS** are – as first order systems – not fulfilling the antecedents of Dunn's theorems. For a first order system supposed to be universal (i.e. modelling its own semantics) questions as to ambiguity arise. Can unintended models be excluded? In a standard setting distinguishing object- and meta-language non-intended models are constructed by *keeping* the intended meaning of all the machinery needed to construct the non-intended model in the meta-language. After dropping the distinction between object- and meta-language re-interpretation tends towards *global* scepticism with respect to meaning. Should we care about global scepticism of this sort?

An ontology of inconsistent objects is – in my eyes – the greatest challenge of/to paraconsistent mathematics and set theory.

Given the strong paraconsistent program of true contradictions and a even mildly realistic theory of truth (containing in some – maybe even restricted – fashion the idea of correspondence), a true contradiction is *supposedly* made true by either an inconsistent fact (taking facts – at least for the moment – to be truth makers of statements) or by inconsistent objects. Like true contradictions they are just *there*.

Mathematics has traditionally been the hallmark of a science that proceeds by proof, and so is free of falsehoods and more so of inconsistency. Changing the basic logic used in mathematics to a paraconsistent logic makes mathematics in a weak sense paraconsistent: If there were to turn up some inconsistency in mathematics, it would not explode. But since there are no inconsistencies expected to arise there, a mathematician will not be inclined to forego the deductive power of **FOL**.

Changing set theory to a paraconsistent set theory makes mathematics paraconsistent in a stronger sense, since now the basic axioms are taken as the inconsistent axioms of naive set theory. There are now real inconsistencies – may be even inconsistent objects – in mathematics and the logic, therefore, has to be a paraconsistent one.

And the inconsistency may not only reside with some elusive set theoretic entities, but there may be inconsistent numbers as well!

To have an inconsistent number theory means at least that within the theorems of number theory there is some sentence φ with φ being a theorem and $\neg\varphi$ being a theorem at the same time. Supposedly this corresponds to at least some object/number *a* being an inconsistent object. Therefore inconsistent mathematics is connected to inconsistent ontology. Its underlying logic has to be paraconsistent.

The problems with having "F(a)" and "¬F(a)" for some object *a* seem not so pressing if *a* is some mathematical object than *a* being a physical object: Mathematical objects are either non-existent – *mere* theory, taken instrumentally – or they are in some elusive Platonic realm where strange things may well happen. If on the other hand one is a reductionist realist about mathematics (mathematics being about structures of reality or mathematical entities rather being concrete entities dealt with by mereology) then inconsistent mathematics is as problematic as your cat being (wholly) black and not being (wholly) black at the same time.

The challenge may not be that great for Naive Semantics given some mildly anti-realistic theory of truth (containing in some – maybe even restricted – fashion the idea that truth depends on justification), and observing that the inconsistent *objects* in that area are *sentences* only. The real problem are objects like the Russell set or the least inconsistent number. Isn't that too much to bear, even for a dialetheist?

The most famous theory in the field is Meinong's ontology of possible, of inconsistent and of impossible objects, and their modes of being.

As a realist – even if you do not adhere to naive realism or extreme versions of metaphysical realism – you adhere to some principle that there correspond structured entities (facts or objects with properties) to true statements. The entities are – at least as much as our linguistic resources to describe them are partly sufficient – as the true statements say they are. This means that a truth like "F(a) ∧ ¬F(a)" means at first sight that the object *a* has property *F* and does not have property *F*. On second sight one will have work with the idea of an extension of "F( )" and an anti-extension of "F( )", the extension being the set of entities fulfilling the criteria of F-ness, and thus being *F*; and the anti-extension being the set of entities fulfilling criteria of not being *F*, thus being *not-F*.

For an anti-realist this may solve the problem of inconsistent objects, since being an inconsistent objects means nothing more for an anti-realist than that the objects fulfils inconsistent criteria. There is no claim on the anti-realist's side that there corresponds something to this in reality.

The anti-realist can even explain how this may happen in case of ordinary objects: If predicates are employed to more or the less vague criteria or family resemblances to some prototype it may happen that one route of resemblance leads from the prototype of *F* to *a*, and another route leads *via* some intermediaries from *a* to a prototype of *non-F*. In the manner of weak paraconsistency one may argue that we have to be able to model theories that depict – at least implicitly – the world as containing inconsistent objects, without ourselves to be committed to this picture. We need the formal tools (like **APS**) for this, but these tools themselves have no negative ontological impact. That is just like we need a logic to draw inferences in works of fiction (or about art) where some works are essentially inconsistent with respect to some object (e.g., some stories about time travelling or drawings by M. C. Escher).

A realist cannot take this easy way out. For (most) realists properties are *structures* of objects – or parts or tropes... – and either you have them or not.

In case of *sentences* – i.e. for a dialetheist view on naive semantics – the way out may be that a sentence is really an object that can have inconsistent properties without us having ontological scruples: A sentence being a dialetheia means that it and its negation are provable. These are clear cut properties. The content of the semantic antinomies, once again, concerns facts *about language*. Given our mild form of realism that incorporates some idea that truth is also – besides aiming at correspondence

– tied to justification we can accept inconsistent objects here, since this 'merely' points to the inconsistent nature of our linguistic access to reality. That is a deep philosophical point – as dialetheism is – but it locates the inconsistent objects somewhere in the objects having the job of mediating between our mind and the rest of reality, these objects often being constituted by linguistic conventions.

Dialetheism in semantics needs no special ontology of inconsistent objects if the inconsistencies are located within our linguistic frameworks. That a sentence can be shown to be true and can be shown to be not true points to the fact of inconsistent evaluations or derivations, but to no deep ontological mystery.

The problem of inconsistent objects is much harder with respect to ordinary objects. If properties are structures of objects, and this means in the last analysis structures of distribution of matter and energy, then an inconsistent objects cannot exist, it seems, since either at some location there is matter or not.

Inconsistent theories in the sciences can be understood in the sense of weak paraconsistency, i.e. they may be *modelled* by **APS**-style quantificational semantics with inconsistent objects, but one need not believe that there *really are* these objects.

You really *need* an ontology of inconsistent objects if you are a mathematical realist and your favourite mathematics is inconsistent, or if you are a dialetheist in a set theory, again taken realistically.

For a dialetheist the problem is naive set theory, given one is a realist about sets. A set, it seems, either is a member of another set or it is not, otherwise the including set could not be well-defined.

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Dialetheist have turned to 'noneism' to have an ontology with inconsistent objects without being either fictionalists or being committed to inconsistent objects in the realist's sense.

So we have to take a short look at noneism here, as this ontological framework distinguishes some paraconsistent approaches from all other set theories considered here. Some short criticism will be pointed at it, but a more comprehensive rejection is part of the broader ontological reflection in chapter VI.

The idea of noneism is taken from Alexius Meinong's *Gegenstandstheorie* (Meinong 1904), his theory of possible, impossible, existing and non-existing *objects*. "Object" in fact becomes a technical term with Meinong. Meinong considers the label "object" as carrying no ontological commitment. In Meinong's ontology *all* objects *are* – in a sense of "are" to be explained. There 'are' also inconsistent and impossible objects. With inconsistent objects there seem to be inconsistencies, since the round square is round, and is square, thus round and not-round. Meinong himself did not develop a logic to deal with inconsistent objects. He saw standard logic as fit for *existing* objects (these are the objects in space and time). Inconsistent objects are for him, because of their inconsistency, *impossible objects* (i.e. it is impossible that they exist, i.e. are somewhere in space and time). Meinong's *basic idea* is the rejection of any inherent connection between having properties (i.e. predication) and existence (i.e. "metaphysics", in Meinong's use of the term). Meinong's point of attack is the *Ontological Assumption* saying:

(OA) Predication implies existence.

and the *Referential Assumption* saying:

(RA) Every singular term refers to some kind of being.

Meinong's more fundamental idea – behind his attack on (OA) and (RA) – is the claim that objects are *beyond being*. That idea is supposedly hard to grasp. Meinong does *not* claim that there are realms of being *besides* being in space and time (i.e. existence). There are no domains of *possibilia* (possible objects). And there are no 'outer-domains' (like in some *Free Logics*) where even more strange objects like the round square reside. Meinong claims that objects *are* without *being*! The philosophical field that deals with objects in this generality is *Gegenstandstheorie [Object Theory]* (whereas *metaphysics* is concerned with existents only).

The central idea put forth by Meinong, going back to his disciple Mally, can be summed up as the *Independence Thesis*:

(IT) Even nonexistent objects have properties and are constituted in some way.

Given (IT) one can truly say of the round square that it is round (as it is square) *without* committing oneself to its existence by this. Predication is independent of existence. Truth does not require referential truth conditions in the ordinary sense.

Meinong's theory has often been ridiculed, from Bertrand Russell (1904) to Gilbert Ryle (1973), who famously claimed '*Gegenstandstheorie* itself is dead buried and not going to be resurrected' (1973, p.255). With the advance of free and many-valued (modal) logics systematic formalisations have been put forth to show its coherence (Jacquette 1996, Lambert 1983).

Richard Routley (1980) and recently Graham Priest (2005) have developed Meinong's ideas as ontological background theory for their dialetheism (their view that there are true contradictions). Routley invented the term 'noneism'. It is a name to an extended theory using Meinong's main claim. Noneism claims, for example, that mathematics does not deal with anything having being. So, for the noneist, there is no problem of *abstract entities*: Numbers etc. are objects and nothing beyond that. The same goes for sets and properties (if taken as *abstract* entities). Noneism is *the* alternative to Platonism, it claims. Noneism is, as well, for its proponents *the* alternative to modal realism (of the sort defended by Lewis 1986) as noneism allows for the same promiscuity of quantifying over any object (of thought) whatsoever.

The theory surrounding these theses Routley sums up in a couple of main postulates of noneism:

(P1) Everything is an object.

(P2) Many objects have no way of being.

(P3) Nonexistent objects have properties and are constituted in some way.

(P4) Existence is no characteristic property of an object.

(P5) Every object has an essence independent of its existence.

(P6) Every object has its characterising properties (in some possible world).

These postulates have some immediate consequences:

- by (P1) anything can be the *object* of a belief, desire, fear – any propositional attitude – whether the object is possible or impossible.
- by (P6), also called the *Characterisation Postulate*, the golden mountain is golden, and the round square is round and is square. The *Characterisation Postulate* is a further claim to the well defined *nature* of nonexistents.
- by (P4) versions of the ontological proof of God's existence should be blocked. As *existence* is no characteristic property all *existence entailing* properties (like *necessity*) cannot be characteristic, and thus are not allowed to use in characterisations.
- by (P3) and (P5) the round square and the triangular square are *different* impossible objects as they have a different set of properties each.
- by (P5) and the existence of *impossibilia* we can say:

  "There is something which necessarily does not exist."

  The quantifier "there is" having no existential impact here.

Not every sentence about a nonexistent object has a truth value, since nonexistents are in most cases only partially characterised. We do not know how many rolls Holmes ate during his observation of the Baskervilles. So a noneist logic may allow for truth value gaps. Priest, accordingly, claims noneism to be the proper theory for intentional and fictional objects.[48]

Prima facie noneism seems a close cousin of metaphysical fictionalism, if not a brand of it. Fictionalism and noneism both maintain – in contrast to instrumentalism, which sees the ontological posits of a theory referring to unobservables *merely* as part of a calculating device for predictions – that the ontological claims made by a theory should be taken as *genuine*, capable of being true or false. Interpreted thus the fictionalist, however, takes these ontological claims as being *false*. Mathematics, for example, is strictly speaking false for the (mathematical) fictionalist, since there are no numbers, but mathematics claims to talk about them. In a similar way modal theories or ethics may lack their supposed subject matter. Despite of this the fictionalist recommends *accepting* these theories, since the aim of theories of this type, dealing with the problematic subject areas, is not providing a true description of how things are in reality, but rather to produce theories with other virtues (like predictive success or the ability to systematize our observations). According to the fictionalist, one may thus not *believe* a theory, since strictly and non-strictly speaking it is false (i.e. not just as a matter of being an idealisation or approximation), but one may nevertheless *accept* it. Fictionalist accounts of this type have gained support and found wider application (cf. Kalderon 2005). Noneism, despite sharing the fictionalists rejection of realism in, say, mathematics and modal ontology and sharing the fictionalist's attitude of taking the ontological claims 'at face value', is almost the opposite of fictionalism: Noneism takes

48 Further on, Priest introduces his noneism in reference to Routley, but tries to improve on Routley's theory (especially on the way to render (P6) less problematic). If a characterisation always holds and characterising an object obeys no restrictions besides excluding existence entailing predicates, (P4), one can introduce for any statement α a characterisation: $x = x \wedge \alpha$. If object b is characterised by that, we have: b = b. Thus with the reflexivity of identity, $(\forall x)(x=x)$, any statement can be derived. Triviality ensues. Priest therefore weakens (P6) to the assumption that there is some possible world at which the properties apply to the characterised object. And one possible world is the trivial world, where everything is true. In line with (P4) the use of "$\Box$" in characterisations has to be restricted as "$x = x \wedge \Box\alpha$" in combination with S5 modal axioms would also trivialize each world! Whether Priest's version improves or waters down Routley's theory will not be discussed in much detail either. Some differences will be commented upon, cf. §5. In any case Priest subscribes to (P1) - (P5) in the presented form, which are central with respect to existence/non-existence claims. He adds his version of (P6). Routley himself later (1995) considered 'items' to be completely free with respect to having all their characteristic features in their 'region' only, so that one may later look how to integrate these regions or make them overlap.

the claims of the respective theories as dealing *about entities*, although, of course, these entities are taken as being non-existent. Correspondingly some of the theories in questions are *simply true* (i.e. true in the same sense observational statements are taken as true by both instrumentalists and fictionalists) if the facts claimed to obtain by these theories are facts about/concerning the respective non-existent entities. Although noneism – *inter alia* – is a theory *of* fiction and fictional entities, the entities it deals with are not taken *as* fiction. Count Dracula is a non-existent fictional object, the number 4 is a non-existent non-fictional object. For the noneist the number 4 is not more fictional than you and me, it just does not exist (i.e. is not in space and time). Routley is very outspoken about this:

> Mathematics is objective; for pure mathematics is concerned with the properties and relations of objects, objects which, though they do not exist, are objective, are in no way mind-dependent … (Routley 1980, p.794)

Fiction presents stories. Fiction is presented in form of mind-dependent human representations. The fictionalist considers the story *as told* as essential in his assessment of an acceptable theory. Not so the noneist: In as much as the mathematical objects are not mind-dependent, mathematics does *not* reduce to story telling and even 'any constructive aspect vanishes' (Routley 1980, p.916).[49]

Further on, noneism, does not contain the problematic distinction between believing and accepting a theory, constitutive for a fictionalist account of supporting a mathematical or modal theory.

Noneism is advertised as solving a couple of central ontological problems or even dissolving some old ontological conundrums. Routley rest his case for noneism on its fecundity to treat a plethora of philosophical problems in a unified systematic fashion.

One major use could be dissolving the problem of abstract entities by recasting them as perfectly harmless. Many statements considered problematic by philosophers (e.g. not only those dealing with abstract objects, but those of indirect discourse, those of fiction and those of false theories) are – even if false in some cases – perfectly in order as they are. The talk about the objects they purport to talk about. If entities like

49 Things are more difficult with Priest's version of noneism, as he takes some mathematical statements (e.g. numerical identities) as simply true and others (e.g. set membership assertion) as not actually true, but true only in those possible worlds where the objects concerned exist (in the ordinary sense of "exist" it seems), although he also allows for these statements being simply true, as they (say again set membership assertions) are not existence entailing. This fits to postulates (P1) – (P5). In any case, he takes 'it to be true that something is the unit set of the null set' (2008, p.209), so committing himself to the non-existent objects (taken as existentially noncommittal 'somethings').

numbers or sets have no existence at all, it does not matter that they are set apart as entities of a special kind (say the subject matter of mathematics).

Sets are objects, there are sets with contradictory properties, but all this, according to the noneist, does not go against our initial realistic intuitions, since sets are not existing entities. Some forms of realism about sets have to be given up on this move, of course (namely those which claim that sets 'are' just in the same way chairs 'are'). Still the noneist may endorse a version of realism that agrees with ordinary (truthmaker) realism on physical objects, and claims for all non-existent objects that true sentences about them correspond to states of affairs containing these non-existent objects and their (non-existent) properties. If truthmaking is *not* understood as a causal relation, a noneist may even consider such states of affairs as truthmakers of the corresponding sentences, adding that these states of affairs themselves are, of course, non-existing objects.

Once the major stumbling block of inconsistent ontology is removed from inconsistent set theory and inconsistent mathematics their case against standard set theories and mathematics is strengthened. Their virtues (like allowing for a strong finitistic arithmetic, cf. van Bendegem 1993, 1999) then may put the standard theories on defence. The viability of noneism, therefore, has wider repercussions than one may have expected.

We return to the issue of noneism when we discuss the broader ontological picture in chapter VI. Note, however, that a paraconsistent set theory may be a Platonist or fictionalist theory as any other set theory. Dialetheism then adds the idea of inconsistent abstract objects or the idea of useful inconsistent stories.

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Apart from its ontology paraconsistent set theories are certainly different with respect to the theorems holding in them. One cannot expect that theorems of **ZFC** carry over, as many of them are proven by means – like indirect proofs – which are not valid in the paraconsistent theories.

One critical instance may be *Cantor's Theorem*. The usual indirect proof proceeds by arriving at the contradiction of the element x_i being and not being an element of the set of the set of the x∈y not being an element of the subset they are coordinated to by the supposed bijection *f* between a set *y* and $\wp(y)$, i.e. $f(x_i)=\{x\in y \mid x \notin f(x)\}$. A dialetheist may simply embrace the contradiction.[50]

Non-existence claims which rest on *Cantor's Theorem* can be rejected by paraconsistent set theories. The set of all truths *exists*: the usual argument for its non-existence does no longer go through (that for any sentence α and any subset *x* of truths the cardinality of questions $\alpha\in x$ exceeds the supposed cardinality of the set of truths; cf. Grim 1991, pp.91-93).

If there was no proof of *Cantor's Theorem* one may well have $\wp(U) \subseteq U$ without further contradiction. The absence of *Cantor's Theorem*, additionally, may wreck also the move to higher infinite cardinalities. Thus the universe U of such a paraconsistent set theory may not resemble V.

In the extreme case – 'extreme' for the common view, of course – we are driven from 'Cantor's Paradise': there is exactly *one* infinite cardinality: infinity. One may embrace this actual infinity and stick to the pre-Cantorian intuition that there come no larger collections than the (simple) infinite ones (i.e. those having this one cardinality $\aleph$).

(NC) and a condition φ similar to the Axiom of Infinity (also mentioning the comprehending set *y*):

$$x = \emptyset \vee (\exists z\in y)(x = z \cup \{z\})$$

open in "x", allow for an infinite set: $\emptyset=\emptyset$ ensures $\emptyset\in y$, enforcing $1\in y$ by the second disjunct, and so forth. The finite ordinals exist by *their* instances of (NC), where the defining condition ψ may simply list their finitely many members (e.g. $x\in 3 \equiv x = \emptyset \vee x = \{\emptyset\} \vee x = \{\emptyset,\{\emptyset\}\}$). The infinite set ω collects them.

If there is no infinity beyond the countable (and thus no properties beyond those which are expressible by the formulas of our set theoretic language) one may use set abstraction to define "∈" by the schematic

$$\text{(D}\in\text{2)} \qquad x \in y \overset{\text{def}}{=} y = \{x \mid \varphi(x)\} \wedge \varphi(x)$$

"∈" thus no longer being the primitive expression introducing set theory; "⊆" being defined in the usual way.

In this way a variant of the Axiom of Constructability, V = L, may return, namely:

$$\text{(Constructability)} \qquad U = L$$

If for all sets there is some defining formula φ the last argument in favour of a distinction between two types of collections, the one defined by a uniting 'rule' and the other merely by its elements (*one* understanding of

50 Even proofs like (Raja 2005) which do not use the diagonalization proceed by reduction, an inference not available in full generality in paraconsistent systems.

the distinction between 'classes' and sets) loses its force. Note again that the arguments against Constructability stemming from the naturalness of the concept of POWERSET and set theoretic realism also have no force in the absence of *Cantor's Theorem*. All subsets can *be there* and *be expressible*.

At the end of chapter I we outlined an approach claiming V to be an entity *sui generis*. We may use the special character of V to account for the viability of **ZFC** then. If one does *not* support such an understanding of V one needs some other (stronger) *formal system* as meta-theory of **ZFC**. So even if set theory is our strongest formal system in applied science, we have to ask where we are when we talk about *it* and V. The main advantage of a paraconsistent approach can be seen in its incorporation of meta-theory into the most comprehensive formal system. A paraconsistent set theory has to have a paraconsistent meta-theory, since otherwise it has to use some theory like **ZFC** again, inheriting all the conundrums the paraconsistent theory was set out to solve. The meta-theory cannot be consistent as it treats of the universe (i.e. treats of an *inconsistent* object), which can only be reasoned about using (restricted) paraconsistent inference rules. A paraconsistent set theory has a model <U,I> U being the domain, I the interpretation function. Their ordered pair is a set, as U can occur in itself and other sets, and so is treated within the very same theory again. And it may be another inconsistent object.[51]

51 If standard logic and set theory are employed in the meta-theory one can use classical meta-theorems to outline (term) models of some paraconsistent set theories (cf. Libert 2003, 2005 and reference therein). This is neither an option in our discussion about U, where pushing the issue in the meta-language does not help, nor in the wider dialetheist perspective, where, for instance, the conditional and identity rules employed by Libert would lead into serious trouble (cf. Bremer 2005, pp.185-98).

VI

BROADER ONTOLOGICAL REFLECTIONS

When considering set theories as ontologies broader ontological issues arise.

On the one hand we have to reflect on ontological methodology. Maybe there are some methodological peculiarities to set theory as ontology. On the other hand we have to face some ontological issues that seem to arise especially with set theory. Foremost this is the issue of an incomplete universe. Also the distinction between sets and other collections has to be considered.

Famously Cantor distinguished sets as consistent collections or 'finished sets' from collections which cannot – on pains of antinomies – be taken as 'finished'. For these 'absolute infinite' collections (or 'multiplicities') Cantor claimed that 'the totality of their elements cannot be thought as "existing together"' (cf. Cantor 1991, p.390). So *they* seem to have *elements* and at the same time the elements do not exist together. This looks like a contradiction – contradicting *inter alia* the Domain Principle – and thus the name "inconsistent multiplicities" seems appropriate.

Another interesting issue in axiomatic ontology is the existence of a null set ∅. Zermelo at least sometimes thought of the null set as a mere technical device. Such an attitude does not square with set theoretic realism. ∅ is the paradigm abstract object. One may be tempted to locate the singleton of the Cologne Cathedral at Cologne, but there is nowhere to place ∅. One may try to avoid using ∅ and demand that every set has a member. Doing so (e.g. Maher 1968) requires to axiomatically introducing some substitute which behaves like ∅ (e.g. in building unions and cuts) without being ∅ (i.e. without being empty the substitute's members do not occur in the unions it builds). This way of employing axiomatic ontology to get rid of ∅ invites many questions for explanations, all of which do not arise in **Z** with respect to ∅ – or can be answered easily. Once one assumes an abstract category of containers it may not look so unnatural to assume ∅. The idea of an empty collection may be more natural than that of a sequence of transfinite cardinals.

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There are problems with a universal set even if the set theory containing it is consistent.

Although one may argue that a universal set is one of the hallmarks of an acceptable set theory, looking at the set theories with a universal set may give one second thoughts.

Set theories with a universal set attack the 2nd objective to be achieved by a naïve set theory: having a universal set. The 1st objective being having Naïve Comprehension. They can be taken as the claim that a universal set can be had without dialetheism. The more general reply to the dialetheist's claim (that we accept true contradictions because we want to see the two objectives of naïve set theory realized) might be that one sees more problems than benefits related to a universal set in the first place.

The first category of objections may be called *negative* in as much as they argue against the supposed gains of a universal set:

(i) Since meaning *isn't* extensional, fixing the meaning of "set" does not require an extension to the predicate "is a set". The meaning of "is a set" is fixed by the axioms governing our identification of sets.

(ii) If the extension of a predicate φ cannot be a set this doesn't mean that there is no referent of φ at all, it just has to be fixed singularly.

(iii) If consequence is spelled out model theoretically one may quantify over all appropriate structures' equivalence classes (in terms of isomorphy). Interpretations interpret languages with countably many formulas. Even sentences talking about uncountable structures cannot be supplied indefinitely. For structures of high enough ranks, even if they do not comprise *all* the sets, it is not obvious that some relevant structure is left out (this would have to be one with no isomorphic structures so far). A finite being cannot scrutinize them anyway.

The second category of objections may be called *positive* as they argue directly against a universal set as violating our intuitions *itself*:

(iv) As the absence of U seems to miss having a set corresponding to *sethood*, **NF** and relatives have no set $[\in]$, $\{\langle x,y\rangle | x \in y\}$, where $[\in]$ is as intuitive as U is, no singleton function etc.

(v) Like "$\in$" "$\subseteq$" could be taken as the basic concept of set theory. Then the powerset axiom is even more fundamental than it is intuitive already. $\wp(U) \subseteq U$ is not only a claim about cardinalities, but claims that *all* subsets of U are already in U. This is counterintuitive, and to block *Cantor's Theorem* some subsets which seem to exist have to be taken as non-existent (e.g. the diagonal set of the supposed bijection f).

NFU 'solves' this problem by having more atoms/pairs than sets, but this – looking at the cardinalities involved – is obscure, because this means just too many atoms or even not every atom having a singleton! The third category of objections asks us to reconsider some ontological alternative:

(vi) Suppose there is no set of all sets but a *realm* set theory is talking of. There is no need to take it as a set; supposedly everything useful can be said talking of sets only. There may be subparts of this realm (e.g. the part where the ordinals are), but again this realm/part-relation may be something like mereological composition. There is no need to have a theory of proper classes (as a theory similar to set theory).

(vii) Once we allow for more than finite sets not every condition defines a set – remember that Naïve Comprehension is consistent on the finite sets! The limits of restricted Comprehension (like in **ZFC**) are just a 'failure' to commit us to even more infinite sets. That doesn't seem so bad. Given the set of finite sets, Naïve Comprehension and (restricted) Comprehension come to the same thing. As Naïve Comprehension means trouble only with infinite sets this again may be rather a problem of the infinite. In the infinite **ZFC**-Comprehension (i.e. Separation) doesn't seem too bad.

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The main alternative to a universal set seems to be the idea of an unbounded sequence of ever higher cardinalities. The idea of classes provides just an intermediate halting place before collecting them to proceed further!

Can we really understand an incomplete universe? The short reply that in reading a book on it, writing a paper on it and having a discussion we obviously understand the position we are attacking is too simple. Obviously one can retell what some authors have published or said about 'absolute infinities' or an 'incomplete universe'. In a sense of "understand" we understand the strangest stories that lack coherence, at least up to a point. Around that elusive 'point' of not coming to terms with a story or theory we are at a loss what somebody is writing or talking about. We are at a loss – we may conjecture – because what we are told goes massively either against our standards of coherence or against our understanding of involved core concepts. This conjecture is still quite vague. And, with respect to paraconsistent set theories, we cannot appeal here to strict standards of consistency against dialetheists without begging too many questions.

Metaphysics in analytic philosophy traditionally (starting with Frege) could be seen as deriving from semantics or an account of the truth of statements/propositions. Frege assumes concepts as entities in their own right and with their peculiar feature of being 'ungesättigt' (having a gap to be filled by some argument), because he needs this ontological assumption in his account of assertions/statements and their cohesiveness. With the increasing interest in ontological questions in their own right metaphysics nowadays if often seen as independent from semantics, not to speak of epistemology. Metaphysics in this newer tradition advances as *axiomatic ontology*. "Axiomatic" is meant in theories of this type in a sense close to the sense of laying down axioms in logic or mathematics. Axioms may be useful (say in applying mathematics in science), but foremost are stipulations concerning either the concepts or the entities contained in them. As Carnap saw 'no morals' in logic, so that according to his *principle of tolerance* every system had a right to be developed, so the axiomatic ontologist may see no morals in ontology so that there are no restrictions on ontological postulates. In the light of this approach one might consider any set theory as just another axiomatically founded ontology. The axioms just being its introduction.

Carnap may well have been wrong about conventionalism in logic, at least in the sense that human reasoners follow one specific logic (or a narrow range of logics) and in that the human language faculty may contain one specific logic (or a narrow range of logics). In a similar way stories and accounts about the possible furniture of the universe may be interesting in their own right, but given even a mild realism, there cannot be just any entity proposed by some story or ontological account. And given, further on, some mild evolutionary account of our cognitive faculties our human conceptual system cannot be neutral with respect to ontological theories.

The debate over incomplete universes vs. inconsistent totalities might thus be put: Does our conceptual system allow for them?

The difficulties which one may have with the mere postulation of some axioms may point to some more general methodological lessons in metaphysics. Something seems to go wrong here with axiomatic ontology. From the point of view of the Fregian tradition, which put semantics before metaphysics, axiomatic ontology has gone too far. Notwithstanding its limits or shortcomings the linguistic or conceptual turn of (analytic) philosophy is still endorsed by a proponent of this tradition. Postulating relations works in science in as much as the postulates are borne out by the thus established (scientific, experimental) practice and theoretical framework. Definitions are not refuted by evidence, but some definitions turn out to be useless or to be incoherent with other parts of a theory. As has been noted often: Conventions (alone) do not deliver truth(s). We understand what it means to tell a story. We may conceive of such a fiction even in terms of images what the world had to be like if the fiction was true. We employ our imagination thus when reading a book others do that when producing a (fantasy) movie. No one, however, can imagine the Russell set or the round square in that way.

To decide between ontological proposals we need criteria to judge their respective merits. These criteria may take up the tenet for which the respective ontology was developed. Alex Oliver (1996, pp.2-13), discussing the metaphysics of properties, proposes to look at an ontology's achievement in conceptual analysis, in as much as ontologies are introduced to account for the function and content of expressions. To understand what a name is one introduces referents, for example. Following such an analytic procedure (types of) entities are introduced which, according to the analysis, account for our use of language. And no other entities are to be introduced, as ontologies have to be economical.

With respect to ontological economy one may distinguish between *ideological economy* and *ontological economy proper*. Ontological economy proper concerns the number of introduced types of entities. Ideological economy concerns the number of undefined basic concepts of a theory. The less undefined concepts a theory has, the more concepts have to be defined, and the more inferential links will be present in the conceptual system of the theory, which therefore shows a higher degree of systematicity (cf. Goodman 1943, 1949, 1972, pp.275-355).

Now, we can increase the ontological economy proper of a theory if we introduce undefined operators and relations instead of basic types of entities (say, if we do not define "necessary" and so forsake the introduction of possible worlds). And *vice versa*: We can define former undefined concepts by introducing new types of entities to occur in their definitions. This is even true of methodological concepts (say, if we substitute "being a relation" by ordered pairs or tuples). How can we decide then the degree of simplicity of an ontology?

> One constraint is the provision of fruitful conceptual analyses of concepts we employ in our ordinary thought and discourse. ... Here, as in all philosophical inquiry, we must adopt the method of reflective equilibrium, balancing the demands of theory against the preservation of commonsensical beliefs. .. [O]ne cannot hope to defend a metaphysical theory by constructing knock-down arguments against each of its competitors. There are numerous ways to trade off ideological and ontological economy and to balance these theoretical benefits against the preservation of common-sense belief. One can only hope to draw up a cost

> and benefit score sheet, it being a very real possibility that there will be ties for first place. (Oliver 1996, pp.4-5)

Systematicity (and thus a higher degree of ideological simplicity) yields more explanatory power, since less fundamental principles (those containing the undefined basic concepts) have to be used. Conceptual analysis provides a decrease in the number of undefined concepts.[52] We explain a fact by conceptual analysis by being explicit about the concepts which are used to describe the fact (cf. Oliver 1996, p.6). So a high number of definitional links increases explanatory power. Therefore, one usually may prefer higher ideological economy over higher ontological economy proper, as regularly theories are chosen for their explanatory power (cf. Thagard 1978, 2000). Thus one will accept those (types of) entities which are introduced in building up the framework of such a theory.

One might consider ideological economy as a *meta-constraint* on ontologies as they are to be embedded into wider theories, which have to compete in explanatory power. Another meta-constraint could tie an ontology to a broader scientific perspective like nominalist scientific realism etc. We will focus here on two basic criteria for ontologies themselves. One may keep ideological economy as an important desideratum in mind. Generally, however, we may prefer the foundations of mathematics to be as neutral as possible with respect to controversies between concrete scientific theories.

These considerations lead to one criterion of a successful ontological theory:

(O1) *Legitimation by Conceptual Analysis*

Exactly those (types of) entities are to be assumed which have to be introduced by the best conceptual analysis of the target domain of concepts.

(O1) has to be supplemented, however, by a second criterion:

(O2) *Epistemological Constraint*

An ontology which introduces (types of) entities has to contain a theory how we know of these entities.

[52] One need not subscribe to a highly controversial theory of concept definitions to have some form of conceptual analysis. The argument presupposes only that some form of conceptual analysis is possible (cf. Jackson 1998), where conceptual analysis is even viable in theories which deny that there are enough definitions providing an analysis into necessary and sufficient features, as long as we allow for any inferential links (cf. Bremer 2008). One may even speak of ‘analysis’ *simpliciter* as one stresses that this analysis yields substantial insights, often denied to ‘mere’ linguistic or conceptual analysis (cf. Williamson 2007).

This constraint will not be endorsed by someone following a 'pure' axiomatic approach to ontology. Giving up (O2) on the other hand means giving up the approach to ontology which ties it to *our* conceptual system and *our* use of language (as any theory of these will have to contain a theory of how we refer or 'hook up to' entities of some kind). Not subjecting a conceptual analysis to the epistemological constraint seems to miss that the target of analysis are *our* concepts and linguistic abilities. Any analysis of our concepts should contain a part explaining how we can *use* these concepts or manifest our knowledge of their proper employment. Even if one does *not* follow the strict agenda of Michael Dummett (1991) and his adherents any theory of our concepts should contain a part explaining our use of these concepts in our engagement with reality and other speakers. If our possessing some concept is reduced to some type of entity (as outlined above) then a comprehensive theory of our concept possession should contain how we can stand into contact with entities of this type or can know of them.

Criterion (O2) seems to stack the cards in favour of some causal theory of knowledge or access. This need not be so. (O2) can be weakened to the requirement of giving an account of how we can *make justified statements* about the entities in question. Taken thus, Plentitudenous Platonism, which claims that we by developing *consistent* theories have justified beliefs *about* abstract entities, passes this test, as noticing consistency is taken as faculty independent of an elucidation of consistency in model theory or logic (cf. Ballaguer 1998, pp.48-75).

David Lewis' modal realism for all its ideological economy, relying just on classes and *all* individuals (everywhere), is mostly rejected by its blatant failure to give a convincing answer to (O2), as Lewis postulates the absence of *any* access connection between us and (other) possible worlds (denying both spatial and causal access). This – as with Plentitudenous Platonism – may be too quick. Lewis may well argue that justified belief in modal realism is enough if there *are* just these worlds.

There may be more conditions one may like to lay down for ontological theories. On the other hand the burden of outlining other conditions for understandable and acceptable ontological theories lies with the axiomatic ontologists.

Let us consider four set theoretical options in the vicinity of universality introduced earlier [in chapters I & V] in face of these conditions. We have to turn to some analysis of the involved concepts.

(1) Noneism

The following discussion will be concerned only with noneism (i.e. with a theory subscribing to the noneist principles (P1) - (P6) *in some form*) and Meinong's thesis of non-being ('Außersein').[53]

David Lewis (1990) complained against Routley's use of two ways of quantifying, one time with existential impact, one time without. The mere occurrence of two types of quantifiers may not be the core of the problem. Non-committal quantifiers have become a common tool in modal and *Free Logics*, usually in combination with an existence predicate. The concern behind Lewis remark on the quantifiers points, however, in the right direction: If "there are" is not committal in *any* sense (i.e. not even to *possibilia*, as *it is committal* in many *Free Logics*), what does it mean after all? We (i.e. we who are not noneists, yet) run against a way of talking defying our understanding. Our first problem is not that we do not agree with the noneist, but that we simply do not understand what to disagree about.

The distinction between existents and *possibilia* is a clear one in modal semantics. Noneism claims even less being than possible being, but still uses the forms of "to be". What an object *beyond all being* is supposed to *be*, is beyond us non-noneists, and our human concept of object, one may suppose. Ryle challenged Meinongianism of abusing the expression "object":

> …the important sounding word 'object' never did have any other positive function than to be a synonym for 'subject-matter' or 'remark-topic' (1973, p.257).

That thoughts have representational content no one will deny, that thoughts have 'objects' beyond those representations is a far more substantial thesis and no obvious consequence of the observation on representational content. For noneism, however, everything depends on Holmes not just being a file of propositions attached to the expression "Holmes", but being a unified something (an 'item'). Noneism seems to trade on the almost imperceptible shift from 'content of a thought' (easily identified by citing the representation employed in the thought) to 'object' (as something *beyond*

53 Thus even if there are major problems for noneism that need not tell against Dale Jacquette's or Terence Parson's theories of non-existents, which they claim go back to Meinong. The same applies to work done in the vicinity of Gegenstandstheorie (e.g. the papers in Haller 1995). In my eyes though, Routley is right in seeing his version of noneism as a proper articulation of Meinong's main theses. Even if that is not so (i.e. if Meinong is misrepresented by the noneist), the challenge posed by noneism is worth considering in its own right. The reference to Meinong then has only a motivating function..

its representation). As Priest stresses: '[A] noneist accepts objects of thought as genuine, not just as linguistic simulacra' (2005, p.42).

The noneist's "there are" posits a "there" which we cannot locate and are not allowed to locate on pains of importing being into the theory. And "cannot locate" means here not just without location in space and time, but without being placed in any 'realm' like Frege's or Popper's 'third world' of thoughts or abstract entities. Chisholm (1973) once appropriately called the noneist's objects 'homeless' as they neither are in the concrete realm (the universe) nor beneath the Platonic forms. The noneist's "there" is equivocal when applied to ordinary objects, which exist some*where*, and non existing objects.

The *Independence Thesis* itself seems, at least in the noneist's reading, quite questionable. BEING-ROUND in case of the non-being round square cannot be the same manner of BEING- ROUND like in the case of a penny coin. BEING-GOLDEN in case of the non-being golden mountain cannot be the same manner of BEING-GOLDEN (i.e. having some physical structure) like in the case of the gold bar. All general terms seem to become ambiguous here!

One may represent states of affairs involving existing and non-existing objects, and properties in the same format, say:

(1) <The golden mountain, Golden, 1>

(2) <Peter's gold bar, Golden, 1>

This, however, is only a formalisation at the level of an ontological theory. What we also need is an account what having a property comes down to. In case of physical objects we have such accounts (like trope theories or property realism). These theories cannot apply to non-existing objects. How, then, are they *to have* their properties?

Again, describing an non-existent object as having some property does neither explain how it has that property, nor does it explain how this succeeds in the object 'being there'. Usually speaking about something does not make it the case. Does this distinction have any application in case of non-existent objects? Even describing a non-existent object other than it was introduced in its characterization need not be a matter of falsity, since non-existent objects may be inconsistent. Even defining a non-existing object as "consistent" leaves it difficult to distinguish false claims about an object from changing the subject or extending the objects definition/characterization.

Thus *saying*

(3) '[V]ery many objects do not exist in any way at all'

(Routley/Routley 1973, p.227).

makes no more truth – or maybe sense – than saying

(4) The Arch-Supervisors co-inhabit collectively the hidden dimensions of Gaia.

Noneists can hardly claim their descriptions and postulates to be acceptable to their audience by being evident. This applies as well to Routley's distinction between *reference* (the supposedly bad idea behind most of today's philosophy of language) and *aboutness* (the supposedly benign and non-committal relation between names and non-existents, *inter alia*). Instead of the *Referential Assumption* (RA) Routley endorses something like an *Aboutness Assumption*:

(AA) Every singular term is about some kind of being.

On first sight (i.e. before becoming a noneist) this is a distinction without a difference. Especially if one holds that reference – as shown by definite descriptions – does *not* require a causal connection, there is nothing that sets *aboutness* apart from reference.

To defend noneism it is not enough to stress that we understand talk about fictional objects and thus talk about non-existents.[54] This understanding only counts for noneism if noneism was the only or best account of fictional objects, which, of course, is contested. We understand what it means to tell a story.

Meinong (cf. 1904, §§2-3) argues in favour of noneism by pointing to (i) thoughts always having 'objects' as content, (ii) the countability even of things we know not to exist, and (iii) negative existence statements. (iii) has been dealt with by Russell's or similar theories of descriptions and elimination of non-referring terms in favour of variables and predicates. (ii) ultimately reduces to (i) as the counting concerns objects of thought. So Meinong's main argument claims that the intentional structure of thoughts (and indirectly sentences) has to be accounted for by positing objects of thoughts, *every* thought dealing with its objects. Meinong thus shares with the so-called 'semantic tradition' (cf. Coffa 1991) the thesis that thoughts have *objective contents*, which can be shared. But whereas the semantic

[54] Both Frederick Kroon (2008) and Daniel Noolan (2008) in their criticism of Priest are not clear on this point. Kroon bases his criticism on a problem with the properties of the fictional character Gandalf, and sees this as 'an example of our apparent ability to talk about what does not exist' (p.199). Noolan says, 'Non-existents are also very useful as possibilia' (p. 191), which for the noneist they are – often enough – not: They have not possible being, but no being at all. Meinong and Routley are perfectly clear on that. Priest employs possible (and impossible) worlds in his version of noneism, which complicates matters a lot, and which invites understanding non-existents as possibilia, as they exist in some 'possible world'. On the other hand he is outspoken against a possibilia account of non-existents, for example: 'I deny that quantification over something requires it to have any form of being' (2008, p.214, Priest's emphasis; similar: Priest 2005, p.14).

tradition sees these contents as sentences or their meanings (i.e. as representations or abstract objects) Meinong uses a generic concept of 'object' and binds it to the *Independence Thesis*. Even if the positing of objects of thoughts was required by a conceptual analysis, say of propositional attitudes, the further step is an addition [see §4 above].

There are other theories of intentionality and intentional objects besides noneism. Making a *de re/de dicto* distinction or employing a *Free Logic* accounts for the failure of quantifying into intensional contexts and related phenomena. Representational theories, which take propositional attitudes either as relation to representation of the *language of thought* or representation of a public language, provide a model of the attitudes without too much ontological commitment. The goal of ontological scarcity or simplicity – if that is a goal to adhere too – thus does not favour noneism over these theories, at least not to a degree which makes dealing with its extravagancies worth while. The crucial – and in my eyes not met – challenge to noneism demands arguments from the noneist which show that in the critical cases of non-existents "thinking of *a* as *F*" is *not* reducible to "having a representation: F(a)" (or a representation belonging into an equivalence class of translations or synonyms of: F(a)).[55]

Noneism engenders epistemological problems both in accounting for reference or the substitute for reference, and for the truth of non-existential committal theories: Are there non-existent truthmakers for facts about non-existents? Frege and other Platonists speak of a relation of *grasping* a thought. Relating to pre-existing thoughts or other abstract objects by such a faculty of grasping is one of the major and controversial topics in the philosophy of mathematics. In this case, however, the problem is simpler than the related problem for noneism. The Platonist has/postulates a 'realm' (a Fregean 'Drittes Reich'), which is targeted by the grasping. Non-existent entities are not anywhere. Their being 'entertained' thus – if not just reducible to the representation of sentences – is far more elusive than Fregean grasping. Meinong himself (cf. 1904, pp.10-12) speaks of 'grasping' a 'pre-given' ('vorgegeben') object, then rejects talking of 'quasi-existence' and settles to the thesis that the contrast between being and non-being arises only with (complete) states of affairs, not single objects, which are, therefore beyond being. This, however, threatens to reduce objects to thoughts about objects in assumptions (presumably sentence like representations); and Meinong's states of affairs ('Objektive')

[55] Priest (2005, pp.58-59) criticises representational accounts of propositional attitudes, without putting weight on these reflections. His main criticisms concern the problem of *quantifying in* and the problem of introducing an equivalence relation on representations without invoking the objects they are *about*. The first may be answered by some *Free Logic* account (with *possibilia*, say), the other by some form of inferential role equivalence between representations.

are (sometimes) said to be non-existent themselves (cf. 1904, p.6)! The non-existence of states of affairs is also very explicit in Routley and Priest. So with respect to grasping a non-existent nothing is gained here. Meinong ultimately declares that *every* object could be known, and everything that *could* (!) be known has 'givenness' ('Gegebenheit', p.20). 'Given where by whom or what?', one inclines to ask. Later Meinong (1921, p.20) stresses that grasping is something 'ultimate, indefinable' ('ein Letztes, Undefinierbares'). The faculty of grasping an non-existent object, so, is a theoretical postulate in Meinong's theory.

Priest also at one point (2005, p.142) posits a faculty of 'pure intention', which is able to bring us into contact with any object whatsoever. He provides no account how this faculty works in detail. Further on, he claims *phenomenological evidence* for acts of object access:

> [W]hen one fears something, one has a direct phenomenological experience of a relation to the object of the fear. And the phenomenology is quite independent of whether or not the object *actually* exists. (Priest 2005, pp.57-58)

As often with phenomenological evidence claims this one seems open to disagreement: Does one not rather experience *the content* of one's fear, whether or not it is (semantically) linked to an object or not? What would it be like to experience an object *simpliciter*, in any case? A representational theory seems closer to the phenomenology.

Priest ultimately tries to soften the access problem by moving his version of noneism closer to fictionalism and RTF:

> The properties of Sherlock Holmes may be just as a *priori* as those of 0. In both cases, we characterise an object purely by fiat. We know a *priori* that the object so characterised has those properties (at certain worlds), and this is so whether the characterisation is provided by what is told in Doyle's novels or by the Peano axioms. (Priest 2003, p.9; cf. 2005, pp.145-48)

In fact, somewhat against the spirit of Routley Priest's noneism is more a fictionalism than noneism. Priest – like the fictionalist – regards fiction as false. Priest also regards it as *possibly true* (i.e. true in the worlds of fiction). If α is any claim about the non-existence entailing properties of a mathematical entity, the noneist should claim α to be *simply true*, whether the mathematical entities are existent or not (as they are not, of course, for the noneist). Priest has to say that α is true at the worlds in which the entities exist, i.e. has to take α as *possibly true*. At the actual world it is true: $\Diamond\alpha$. Mathematical statements become intensional. Priest so endorses a revisionist theory not only of mathematical ontology, but of mathematical truth as well. If that saves noneism then it does so only at the price of moving towards fictionalism and trading in the problems of fictionalism. As Priest uses the strategy of entities at some world not just for mathematics his theory – despite (Priest 2008, p.214; 2005, pp.14, 42) – looks rather like a theory of *possibilia* in a paraconsistently extended *Free Logic* than like Routley's or Meinong's theory of non-existent items.[56]

The criterion (O2) of coming into contact with entities of a proposed type seems questionable if an ontology itself entails that there is no (causal) contact to these entities. Noneism here resembles those versions of Platonism which deny that abstract entities interact with the physical world. As noneist 'items' are not existent requiring access seems requiring too much. But this is the basic problem with noneism again: 'items' are said to be 'there', thus are not nothing, or – again – the whole theory seems incomprehensible. A fictionalist invents stories and thereby accesses fictions easily. Noneism claims to be something different from fictionalism, but it seems it cannot.

Noneism is incompatible, further on, with any form of semantic externalism, as externalism typically invokes some causal mechanisms of hooking up concepts and lexical items to properties in the world. Noneism's attack on the *Referential Assumption* has to apply to properties and 'content' as well. As many epistemologists and semanticists support at least some version of externalism noneism calls for a far and wide revision of these fields as well, respectively is put into doubt by the success of externalist accounts. Externalism has its own problems with fictions, but any progress here helps fictionalism – again – and not noneism.

Noneism in denying being to inconsistent objects is too weak as foundation of paraconsistent set theory, since sets – unless you believe in noneism – are usually taken to exist (even if not in space in time). Noneism certainly is no version of realism.

Noneism is either a version of a *substitutional* understanding of individual expressions such that we can use a non-referring expression and *tell a story* in which this expression occurs (for example a story about a unicorn) in which case intensional contexts are to be understood in some way paratactically in the tradition of Carnap's *intensional isomorphy* or Davidson's *saying that* – or *noneism is not comprehensible at all*.

56 There are a few additional problems with circularity for Priest's account, it seems. Firstly, possible worlds are considered to be non-existents as well. So what can it mean that *all* non-existents exist at some world? Priest's version of *Characterisation* has worlds 'realize' other non-existents and their properties. As worlds are taken as non-existents, *where* do *they* get realized? Or can they realize without being realized themselves? Worlds seem to exist at themselves. Secondly, discussing the question how non-existent mathematical objects can be used to describe reality (the actual world) Priest claims that the physical quantities and the mathematical quantities 'have the same structure' (2003, p.12; cf. 2005, p.150-51). But a structure is a paradigm case of something abstract, thus non-existent for Priest. A regress seems to ensue.

(2) Incompletability

One of Cantor's arguments in favour of transfinite numbers was the naturalness of extending the successor function and the taking of unions beyond the finite. Once one has understood that the finite sets as a collection have to be taken in a set as their collection, one naturally extends the order ω of this set to $\omega+1$, $\omega+2$… Who admits the ascent to *ever larger* natural numbers cannot resist the ascent to ω and above. (Therefore strict finitism attacks already at the first ascent.)

The picture behind the iterative hierarchy seems to contain the idea of *indefinite extensibility*. Whatever axioms of large sets we add we can imagine the 'process' of set generation going on from there beyond these intermediate halting points. Incompletability seems to be built into this picture of sets. No collection provides a natural stopping point to any further 'construction'.

Can we comprehend incompletability? Properly speaking – as we are *not* talking about processes of reasoning or construction, but about ontic structures: Can we comprehend an incomplete universe?

The concept seems to be beyond comprehension as it is logically true: Everything that exists exists. Whatever is in the hierarchy *is*. Thus – we seem to be forced to go on – it is *somewhere*. Thus this *somewhere* cannot be incomplete.

The concept which causes these conundrums is the concept of quantification. Once we subscribe to the idea that quantification requires a domain we are set for trouble. But how could QUANTIFICATION and DOMAIN OF REFERENCE ever become decoupled? Does our conceptual system allow for – or even contain – a concept of quantification *without domain*?

Raised in standard meta-logic we are accustomed to assume not only a domain, but we see this domain as another *thing*, which then 'of course' has to be contained in *its* (i.e. *another*) domain.

The INDEFINITE is a crucial concept in Hilbert's finitism. Speaking of an indefinite object seems to allow having an object of arbitrary size without committing oneself to quantifying over a domain of infinitudes. Hilbert also invented and used his ε-operator to this purpose.

Hilbert's finitism (Hilbert 1925) was not directed against the idea of the ever larger infinities of 'Cantor's paradise', but was inspired by the idea of secure foundations for talk of infinity, where in these foundations (considered as meta-logic) the notion of the infinite was not to be presupposed. What is interesting even for the critic of infinities in Hilbert's finitism is his method of trying to work around the commitment to actual infinities (cf. George/Velleman 2002: 147-72; Shapiro 2000: 158-65).

Hilbert wants to justify set theory and Cantor's theory of transfinite numbers by using only *finitary arithmetic* in the meta-theory. Finitary arithmetic includes equations and their truth functional combinations. Also sentences with *bounded* quantifiers, like "$(\forall x<120)$", are admissible. Any combination of such sentences is effectively decidable (by dealing with finitely many specific numbers and their properties). Now, to include some generality (i.e. to be able to make general statements like the commutativity of addition) Hilbert introduces *schematic* letters: *a, b….* One can thus express

(CA) $a + b = b + a$

Hilbert considers a statement like (CA) to be finitary! The idea is: Which ever specific numerals we choose to replace "a" and "b" the corresponding statement will be an acceptable (decidable) finitary statement.

Whereas in standard logic one typically reasons

(UG) For some arbitrary x: $F(x)$

…

$G(x)$

Thus: $(\forall x)(F(x) \supset G(x))$ *since x was arbitrarily chosen*

the finitary reasoning is different. (UG)-like reasoning infers to *the totality* of the domain. Finitary reasoning rather argues:

(FG) The following proof *scheme* is valid *for any instance*:

$P_1(á)$

…

$P_2(á)$

Thus: $P_1(á) \supset P_2(á)$

Nothing is supposed about a totality of objects. It is rather provided a scheme to turn the assumption $P_1(á)$ for any given or thought of individual term into a proof of $P_2(á)$. One might express this as the dialogical challenge 'Once you name the object, I will provide the proof that it is well-behaved as well.'

Employing the Wittgensteinian distinction between *saying* and *showing* one can understand the distinction between (UG) and (FG) as having (UG) *saying* what (FG) only *shows*, where, of course, Wittgenstein, who after 1929 took a position close to the use of schemata by Hilbert or Skolem (cf. Marion 1998), would add that what (FG) shows cannot be *said* at all, since there are *no* completed infinities.

Since no totality of objects to be quantified over is presupposed one need not assume even that there is a totality as the infinite set of natural numbers. One just claims that theorems can be proven for any number expression that someone comes up with. Since this is a general claim about forms/schemata of theorems – what else should one wish for in arithmetic?

Apart from the difficulties with Hilbert's larger program (like the incompleteness theorems, especially *Gödel's Second Incompleteness Theorem*) finitism in this sense has its own difficulties. What to think, for example, about the negation of schemata like (CA)? One might think they are equivalent to sentences or schemata with an *unbounded* existential quantifier. Hilbert thus sees them as 'transfinite propositions', i.e. as illegitimate in finitism.[57]

Nevertheless one may take up the idea of schematic representation. Do we have some additional faculty of schematic abstraction? That needed some

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[57] And even excluding negations of schemata like (CA) seems to leave one with *primitive recursive arithmetic* (cf. also Tait 1981). Remember that the *primitive* recursive function do not include μ-minimization. Primitive recursion comes down to bounded quantification. To have the usual means of logic available Hilbert allows these non-finite formula in, but considers them as 'ideal' (i.e. devoid of respectable finitist content). Formulas are only considered in their inferential role. The formulas themselves can then be taken as the new (material) objects of this reasoning. Thus finitism gives way to *formalism*.

Three side remarks may be in place here. The first concerns strict finitism as a radicalisation of finitist ideas. In case the universe is *complete* there are two options: it is finite or it is infinite. Most set theories we have considered – in any case all used in mathematics – are committed to many different infinities. A universal set within any of them or an updated version building in U has U as infinite set. For the strict finitist, on the other hand, U as well as all other sets have to be finite sets. This leads to some more peculiar features of strict finitism (besides, say, its limitations of self-reference by having limited resources of coding).

The second side remark concerns the idea of indefinite extensibility in relation to *Gödel's Incompleteness Theorems*. By *Gödel's Incompleteness Theorems* we know that unfortunately standard consistent set theories are negation incomplete in arithmetic. When we start with axioms of larger cardinalities we establish stronger set theories. As stronger theories can prove the consistency of weaker theories we have in a theory $\mathbf{ZFC}^{+}$ some theorem stating the consistency of **ZFC**. This theorem – by way of Gödelization – has to be an *arithmetical* truth not derivable before. Thus even standard arithmetic seems extensible by this procedure. The extensibility of arithmetic, however, cannot be indefinite. There are not enough arithmetical truths. If an arithmetical property is modelled by a number set there can only be $\aleph_1$ many questions: for $\aleph_0$ many numbers we can ask whether they are in one of the $\aleph_1$ many sets. Arithmetical extensibility is limited. If we add indefinitely many axioms of large cardinalities the preconditions of Gödelization finally give in as well.

Thirdly, one should not confuse the incompletability envisaged for the 'universe' of sets with 'productivity' in computability theory: productive sets can be extended, but as this extending concerns recursive enumerability the extended sets have a definite cardinality in the lower infinite.

explanation. The ineffability problem raises its head. Postulating a basic faculty of schematic intuition deserves – at least – not more credit than postulating a faculty of intuiting V. One may ask in general what understanding a schema – especially one over an arbitrarily extensible ‘ream’ of *individuals*, not over countable many well-formed formulas of a specified language – comes to. Is it not just to understand that some schematic representation is true/well-formed/valid for *all* its specifications? In understanding the schema we seem to have access to the *domain* of its instances (respectively the domain that these instances are talking about). In this case we seem to have a strong intuition in favour of some Domain Principle.

Dropping the universal quantifiers and reversing to substitutional schema or rules instead of axioms in set theory does not help either, as there are neither enough names around for substitution (names being countable anyway) nor do we get rid of embedded universal claims so easily: just look at the Axiom (schema) of Replacement.

So, the idea of an incomplete universe does not far well in face of condition (O1): our concept of quantification seems to be tied to a Domain Principle, which cannot be circumvented by talk of the indefinite or schemata. The incomplete universe resists stable quantification. If the universe is extensible and grows – between different times? – universally quantified set theoretic sentences shift in meaning – at least in their truth-conditions – as their domain of reference shifts!

Another way to get rid of or ignore the Domain Principle may be to rest one’s understanding of quantification *solely* on the semantic rules for quantifiers (i.e. introduction and elimination rules), similar to so-called ‘anti-realistic’ theories of meaning (cf. McGee 2006). Semantic rules may be (partially) constitutive of meaning, at least for logical vocabulary, explicitly definable. Understanding the rules for “∀” and “∃”, however, involves somehow – it seems to me – thinking of the source where the instantiation terms or assigned values of variables come from, i.e. a corresponding *domain*.

And even if we drop the Domain Principle we have not made the idea of a thought independent but at the same time incomplete universe of sets any clearer. In the context of set theory no *further* objects can *come* into existence. The status of the universe itself as an object of incompletable nature awaits elucidation. The whole idea has an radically anti-realistic flavour, foreign to the set theories discussed here.

(3) V as entity *sui generis*

Given the difficulties in understanding an incomplete universe and the fundamental role of a Domain Principle, why don't we just talk about V without assuming it to be the value of a bound variable? This appears reasonable as doing otherwise land us in an incomprehensible framework of indefinite existents.

Assume we do not give up on the infinite, whether we are Platonists or fictionalists or whatever else. There are then infinite collections. Comprising within them *all* of a kind not collectable itemwise by finite beings. We collect them using our concept COLLECTING. If we talk about the *F*s we naturally assume that there is a collection *F* where they are in. Cantor's Domain Principle expresses this idea that the *F*s we quantify over or talk about can be collected into a totality. Sometimes the totality has to be of another type to avoid antinomies (e.g. in the set/class-distinction). As there seems to be no limit to this procedure we always progress to a wider domain.[58] The Domain Principle thus enforces the idea of the incomplete universe. A domain is added to the objects, giving a larger domain, which is added to the objects – and so on.

Unless, that is, we meet a fixed point in this progression. Informally, the totality of things to be thought of or to be talked of can be thought or talked of: it belongs to the very domain it defines. Thinking of 'the domain x is in' applied to it leaves us at it. Thus it may be called a fixed point of the Domain Principle.

Is U of this type? Having $U \in U$ requires several other adjustments in set theory. And they do not come cheaply – up to inconsistency.

As intuitive as the Domain Principle may occur to us, leading us up the ladder of the indefinite may be too much, as we have seen above. We might accept that the whole construction has a limit: a collection *beyond* further collecting. There lays the naturalness of Limitations of Size: There is *one* size too big to be collected into a set. This collection better not be the set U to avoid severe complications in set theory, otherwise rather intuitive. So one may see the idea behind Limitations of Size *without* endorsing **NBG** or **MK,** or any other set/class-theory. Nothing is gained by having (several) classes. With a collection of classes the question of their collectability immediately arises.

The single limit object V might be different.

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[58] Recently Rayo and Williamson (2003) and others have argued for 'unrestricted First-Order languages', i.e. for quantification *without* a domain. The formal proposal, however, must employ **SOL** and a richer meta-language for which similar problems arise. One may also consider the employed **SOL** as critical and problematic (cf. Weir 2006). Unrestricted quantification is only unrestricted beyond an object/meta-language distinction.

If that limit object V exists – neither a set, nor an extended set like an inaccessible cardinal, nor a class – **ZFC** is consistent. And if our intuitive notion of set rather endorses the General Continuum Hypothesis we add it as well: **ZFGCH** is consistent, if V exists. Our notion of set suits V, and *vice versa*.

This conception of V as collecting all the sets but being a special limit object may correspond better to our concept SET than taking set theory just as the realm up to the first strongly inaccessible cardinal. [The Cantor quote setting the theme of this book may illustrate this perspective.] Someone might argue that our concept SET takes us thus far, but that there are other mathematical objects and theories (especially those of large cardinals, measures etc.), which pick up the baton where **ZFC** hands it over. Although this sounds like a nice division of labour, the large cardinals are too set-like to provide a natural boundary to our concept SET, supposing it to fit to **ZFC** in the first place. V is a stop point, the first inaccessible cardinal is not. And large cardinals – again – give rise to the question where their hierarchy is collected in, inviting and requiring V or some V', landing us again in an incomplete universe. V is not an incomplete universe at all: although we cannot walk, count or 'powerset' us up to it, V contains all sets; they are not in the making, there are no processes of indefinite extension going on. In this respect V as an object at the limits of thoughts differs from the row of experiences discussed by Kant in the *Critique of Pure Reason*: Kant traces the antinomies to their common error of taking the series of experiences, which is only given piecemeal and prospectively ('aufgegeben') as a 'given' totality. As experiences are obviously under temporal construction their series can never be united – by whom? In an experience? Sets, in contrast, are not (temporally) constructed and thus should be collectible in a unity. Thus far we are carried by the Domain Principle. At that *limit* we 'simply' have V as an object, and stop adding it to a domain.

Our concept of SET may force stronger set theoretic axioms on us. This shows, however, not the incompleteness or growing extension of V, but the incompleteness of a theory like **ZFC**. Urelements and ∅ have no members, but are members; sets have and are members; V has members, but is not a member: it occupies a slot in conceptual space.[59]

In the light of the two criteria (O1) and (O2) we may say: our idea of V is an idea contained and connected to our concept SET. The special nature of V is forced upon us by the unfeasibility of the idea of an incomplete or thought independent but growing set theoretic universe. We know of V by

[59] The slot of neither having members nor being a member finds no existing filler, if there is not David Lewis' atomless, uncollectible 'gunk'.

the picture we have of the iterative hierarchy and the structural relations between the ranks.

This conception of V follows some intermediate path between the two ontological traditions in analytic philosophy. On the one hand there are reasons of conceptual analysis why V suits our concept SET. On the other hand some peculiar postulates need to be laid down for V. “V” is a rigid designator naming an entity which does not belong to some domain of quantification, although all other entities and referents of names do!

The major difficulty here would be to allow for a level or form of meta-theory when talking *about* V which is outside of any *formal* system. That way may lay ineffability or some version of ontological semantic mystery!

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Comparing noneism, the incomplete universe and the thesis of V being an entity *sui generis*, the third idea comes out best in its combination of conceptual analysis and axiomatic ontology.

If we do not want to follow this path and acknowledge the problem of talking *about* V outside of **ZFC** a paraconsistent *universal* formal system seems to be obligatory. A system that integrates theory and meta-theory.

(4) Inconsistent Platonism

As mentioned before Ballaguer (1998) defends Plentitudenous Platonism as an option in the philosophy of mathematics, since Plentitudenous Platonism can meet both criteria (O1) and (O2). Platonists like Frege in his *Grundlagen der Arithmetik* (1884) argue at length that Platonism is more natural and more embedded into our conceptual intuitions than any rivals like naturalism, psychologism or constructivism. Although we cannot get into direct (causal) contact with abstract entities if we possess no faculty of logical intuition, we can have beliefs about them which may turn out to be true. If *all* consistent abstract realms exist *and* we have an *independent* grasp of consistency, we have justified beliefs which are true. Ballaguer argues in favour of an *independent* sense of consistency of a (formal) system or construction: noticing consistency is taken as a faculty independent of an elucidation of consistency in model theory or logic (cf. Ballaguer 1998, pp.48-75).

Now, consistency can, of course, not be required of paraconsistent constructions or formal systems. These system and constructions, however, employ *non-triviality* as their substitute for consistency. A system has to be shown to be non-trivial, even if inconsistent. One can thus endorse the position of *Hyper-Plentitudenous Platonism*: all non-trivial abstract realms exist. This Inconsistent Platonism inherits all the virtues of consistent Platonism (i.e. the arguments concerning the applicability of mathematics etc. can be used here as well). One has to claim that we possess an independent faculty of discerning *non-trivial but nonetheless coherent* (i.e. systematic and explanatory) formal constructions and systems. I don't think that this is much more than is demanded by Plentitudenous Platonism. The whole development of paraconsistent logics in the last 35 years bears witness to this. In fact one may assume that we even more easily grasp non-triviality than consistency. Therefore Inconsistent Platonism or Hyper-Plentitudenous Platonism is an option!

The theory fits paraconsistent set theories and circumvents the troubles noneism runs into. Object talk and quantification are understood as always. They only – but obviously tremendous – difference to the common understanding of sets as abstract entities is the thesis that there exist many, many inconsistent abstract entities.

Inconsistent ontology is hard to swallow when talking about objects in space and time: How can it be that there is at some location in space time some atomic structure and yet another or the absence of the first? For abstract entities this is different: possessing two inconsistent properties means possessing two properties so that one of them implies that a sentence negating the possession of the other is true of the object in question. This makes a contradiction true of the objects in questions. Dialetheists endorse such contradictions. They argue that *having a proof* for each side of the contradiction no more can be required in standards of *justified* belief. As abstract entities are not in space and time – or at some other *place* – the incomprehensibility of possessing two contrary properties simply does not apply. Often we are endowed with criteria for attributing *F* and another set of criteria for attributing *non-F*. If these independent routes of justifying a property ascription lead to a contradiction, then so be it for the Dialetheist.

Inconsistent Platonism in addition to meeting (O2) like the Platonist and inheriting some general arguments for Platonism meeting (O1) brings with it further resources in meeting (O1):

- all those intuitions in favour of the existence of U (i.e. that our concept of SET entails the existence of a collected unity of sets),
- all those against a set/class-distinctions (i.e. that we have *one* basic concept ABSTRACT COLLECTION, which is not divided into SET and CLASS, a distinction introduced only be the needs of some theories in mathematics),

- all those against the incomprehensible idea of an incompletable universe (i.e. in favour of the thesis that QUANTIFICATION is tied to a Domain Principle),
- all those against special objects we can talk about, but at the same time have to exclude from a broader domain of quantification (i.e. against a liberal use of axiomatic ontology to ensure a special status for V).

Thus Inconsistent Platonism is a live option, it seems. The main difficulties with Inconsistent Platonism are also obvious: One may not like abstract entities in the first place, and one may not like inconsistent ontologies at all.

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If an inconsistent or noneist ontology is too much to swallow when taking on such a paraconsistent system, then we have to opt for at least partial fictionalism with respect to (some) entities proposed within paraconsistent set theories. Then the exploration of universality in set theory naturally awaits a further thorough exploration of fictionalism. Too many difficult questions wait there: Fictions like fictional characters in literature depend historically and genetically on their authors, and maybe on still existent copies of the literary work and living readers (cf. Thomasson 1999), nothing of this sort can be said of pure sets. Pure sets (like in **ZFC**) are not just presented as abstract entities outside of space and time, but their presentation (the story told by **ZFC**) arguably does not depend on any particular set theorist – not even Cantor, Frege or Zermelo. There might be several intermediate ontological categories between such purely abstract entities and spatio-temporal entities (cf. Thomasson 1999, pp. 120-33). Even in **ZFCU** one may wonder about the singletons of contingent urelements like the Cologne Cathedral: It seems bizarre to assume it to exist before the building was finished or even planned, thus this *set* seems to have a historical place! The recent interest in fictionalism may lead to increased ontological options – but don’t hold your breath!

Abbreviations and Notation

Standard symbols are used: $\forall$, $\exists$, $\in$, $\cup$, $\cap$, $\supset$, $\subseteq$, $\wedge$, $\vee$, $\top$, $\bot$, $\times$, $\neg$ …

$\wp(x)$	is the powerset of *x*.
$\lvert x \rvert$	is the cardinality of the set *x*.
$\rightarrow$	is a relevant conditional
$\Rightarrow$	is used in rule statements

Greek letters φ, ψ … are used as schematic for unspecified formula or predicates (open formulas) of a given language. Γ, Σ, Π are mostly used for sets of formulas.

CAPITALIZATION is used to signal that we now talk about a concept.

Single quotation marks are used in quotes and as 'scare quotes' of established theoretical terms. Double quotations marks quote an expression. Names of famous theorems are *italicized*.

M	is the set of urelements
U	is the universal set
V	is the complete iterative hierarchy, V_α a rank in the hierarchy

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