Combining Realism in Logic with Anti-Realism in Mathematics

This paper tries to combine a version of realism about logic with a version of anti-realism about mathematics. Whether such an unusual combination is viable seems worth exploring, as usually anti-realism combines a – with respect to standard logic and mathematics – revisionist approach to logic and mathematics (typically in some form of constructivism). The view developed here denies the very idea of 'revision of logic' (in some sense to be explained) and takes a structure like ZFC as the backbone of (pure) mathematics. Alluding to Carnap's famous *Principle of Tolerance* it claims that there is no room for tolerance in logic (in some sense to be explained), but a lot of room for tolerance in (pure) mathematics.

§1 When scientists get into trouble with their theories it is theories which are revised, not reality. If your biological account of an organ, say the kidney, provides no coherent explanation of the data you cannot revise the kidney, your account of its structure and function has to adapt. The same applies to the organ brain ('mind/brain' as is sometimes said).

Seen from this perspective the very phrase "revision of logic" has a misleading tone to it. Compare the case of languages: You can chose to talk German if you are able to when doing business in Germany; you can chose to speak Esperanto to impress your peers; but you cannot chose to have no natural language at all. Despite differences in approach and detail linguists agree that humans possess a language faculty, which is uniform species wide. The mind is not a blank slate. The language faculty has an initial state containing principles and parameters to be set. From this perspective (you may call it the 'cognitive science perspective' or the 'Chomskyan perspective') the same applies to logic. Humans possess – besides or as a part of – the language faculty a logic faculty or module that comes with a certain structure of principles. This structure is as it is, there is no room for 'logical pluralism' here. Theories of logic share the fate of linguistic theories: they have to be revised if incoherent in face of the data. Theories of logic are revised, logic isn't.

§2 The questions to be raised now are: 'Are theories of logic in any way different from ordinary theories in cognitive science?' and 'What are the dimensions on which theories of logic should be evaluated?'

A crucial difference might be that logic is considered to be normative. Following proper logical rules helps to infer true (or in any other way designated) statements from the other true (or in any other way designated) statements. Some rules may be more appropriate in some contexts than other rules. Thus we come to see some formal system ('a logic') to be used on some occasion and not on an other. Logic thus seems up to choice. Call this the 'logical positivist' or 'Carnapian' perspective on logic.

Choosing logic cannot be regarded as the whole truth for the simple, but fundamental, reason that in choosing a logic the mind cannot be a blank slate. Some core principles have to be operational in deciding on an applied logic. This core may be the logic faculty. Further on, normativity does not stand in conflict with explanatory theories. Compare linguistics again: Norms do not cease to be norms just because you describe their structure and give a (coherent) account of their function and what following them achieves.

Thus there is room for Carnap's *Principle of Tolerance* in choosing applied/regional logics, but behind and besides this we can study the core logic of the logic faculty.

§3 After these preliminaries we have to turn to the dimensions on which theories of logic are evaluated, and what reasons may be given to prefer some set of rules to another set (which in terms of 'revision' can be read as: what reasons can be given to revise standard FOL).

In analogy to the general philosophy of science we have to look at the issues:

(1) What are the data a theory of logic has to account for? (To be considered are the issues of 'intuition', 'access', 'psychological reality' and reflective equilibrium.)
(2) What are the criteria of better coherence in case of a theory of logic?

Paradoxes can be considered as a *heuristic* to assess the coherence of a theory of logic, respectively its accompanying set of rules/axioms. A paradox/antinomy shows that a set of rules/axioms is not maximally coherent, has limited application.

In addition to meeting some standard criteria of coherence a theory of logic has to meet further criteria as being part of a comprehensive theory of cognition, like

(3) feasibility (of the set of rules in complexity measures)

(4) being embeddable into a wide (partially evolutionary/naturalistic) theory of cognition (which raises, for instance, the issue of evolution going for working solutions in standard environments, not for principled solutions).

Lastly a theory of logic has to

(5) relate logic to epistemology and logic's function with respect to achieving epistemic virtues.

§4 What are the data for a theory of logic? On the one hand we can observe how people reason. Collecting examples and generalizing – maybe by disregarding supposedly obvious errors – one may thus come to a corpus of somewhat idealized ordinary argument patterns. (Generalizing and idealization are not completely unproblematic here, but no more than in other areas of science.) On the other hand a theory of meaning (for logical vocabulary of just for words in general) will come with a set of inferences based on meaning, thus being logical. Bringing these two sources together Nelson Goodman in *Fact, Fiction and Forecast,* John Rawls in *A Theory of Justice* and others have developed the idea of (wide) *reflective equilibrium*. The equilibrium has to take our intuitions of validity into account. Given a re-construction of the inferential rules and meanings (of logical vocabulary) involved, some of these pre-reflective intuitions can be superseded. Paradoxes (like the 'paradoxes of material implication') and antinomies (provable contradictions) play the role of abnormalities and recalcitrant data. A theory that can explain them away or accept them scores higher on the observational requirement of meeting the data than theories which do not.

Preferably capturing the rules of logic in some area of reasoning aspires to the following two ideals:

I. *Intuitive Correctness*: The inferences underwritten by the logical systems are intuitively valid.

II. *Intuitive Completeness*: All the inferences considered to be valid intuitively can be derived using that logical system.

Within cognitive science the ideal of reflective equilibrium has be extended to the idea of *wide reflective equilibrium*: One has to consider not just our intuitive judgements of validity, but also constraints of cognitive (computational) complexity and learnability (in a social or evolutionary context).

Another important constraint concerning the data basis of logical theory is

III. *Accessibility:* All inference principles of the logical systems have to be cognitively penetrable.

In contrast to syntactic principles in linguistics, which are often or mostly processed sub-doxastically, rules of inference have to be accessible to rational agents and speakers to some degree (have to be 'cognitively penetrable'). Rules of inference are employed and appealed to in communication and deliberation. Justifying assertions involves in principle the appeal to inferential procedures and standards of argumentation. These cannot be completely beyond the ken of the agents/speakers participating. Thus a logical theory postulating inaccessible principles can be ruled out. A logical theory containing gerrymandered or highly complex principles we cannot understand on first hearing is at least put in doubt.

§5 General philosophy of science adds to the *observational requirement* of a theory fitting the data, preferably all the data (captured in the requirement of 'data completeness') criteria for a coherent structure of a theory. We consider here: *simplicity*, *explanatory power*, *consistency*.

§5.1 *Simplicity* comes as ontological simplicity and as methodological or structural simplicity, which is equivalent to explanatory power.

Ontological simplicity may concern either the number of types of entities allowed for in a theory or the number of entities (of some/any kind) allowed for in a theory. In the case of logical theories a contentious posit are possible worlds. Possible worlds have become common parlance in semantic model theory. One lesson to be learned here may be: As competing logical theories all employ possible worlds they are in the same boat with respect to that measure of coherence; criteria of coherence (and theory choice) can be indecisive in face of our best theories if they share the features related to these criteria of coherence; but if one theory stands out from the crowd of its competitors in that feature the scales can be moved in its favour.

Back to possible worlds: Suppose – unfortunately contrary to common practice – that the theories which employ possible worlds are clear about what they mean possible worlds to be. If "possible world" is only a title for some set theoretic structure we have only a case of wrong advertisement. The interesting case comes with the assumption of possible worlds as entities *sui generis*. In that case there seems to be an argument involving ontological simplicity available: Most theories that employ possible worlds already employ abstract entities (like sets). Models are set theoretic structures. If models are around anyway then models can stand in for possible worlds. Ontological simplicity decides in favour of models and thus against possible worlds. The argument can only be toppled by an appeal to explanatory power (i.e. that possible worlds are needed to explain semantic or logical features unexplainable otherwise). Typically (with the exception of David Lewis in *The Plurality of Worlds*) such arguments are missing.

Simplicity in the number of entities seems to be unimportant as most logical theories allow for an infinity of entities anyway. This need not be so, however, if some version of finitism can be sustained. The supply of expressions of a logical system need not be endless, but may be indefinitely large, so that in all practical employments of the system we never run out of expressions. If there are not infinitely many numbers (or what not else) then the logical meta-theory can employ finite set theory instead of standard set theory (like ZFC). Apart from dealing with finite collections only finite set theory has also the explanatory advantage of containing absolute complements and a universal set.

§5.2 *Simplicity in explanation* (mostly considered as '*explanatory power*') is the key criterion of theory choice. A theory with simpler principles has more explanatory power as less or simpler principles cover the same ground as more or more complex principles do in other theories (given, of course, both theories fulfil the observation requirement). In case of logical theories theories involving less principles/rules or reduction sets (of logical vocabulary) may thus be preferable to those more complex. An interesting debate around that issue may be Michael Dummett's case for intuitionism. Dummett claims in *The Logical Basis of Metaphysics* that the intuitionistic rules for logical junctors and quantifiers are more appropriate than the standard rules as the intuitionistic introduction rules (in natural deduction) match the elimination rules; he states his case for some 'harmony' between these rules (some 'Harmony' with capital 'H' some 'harmony' without) as they are independent of each other, thus the rules for negation conservatively extending conditional logic, and so on. Dummett tries to establish 'harmony' as a new criterion to prefer a logical theory – as simplicity disfavours his account: Propositional logic can be reduced to a single logical junctor (say the Sheffer stroke). That one junctor allows to derive the complete set of propositional junctors, thus covering the maximal ground. It is much simpler to assume that the logic faculty comes equipped with the Sheffer stroke than to assume a set of junctors each independent of each other. Dummett appeals to theories of learnability, but an appeal to evolutionary theory may outweigh that: We can easily imagine that evolution equipped a cognitive system with the capacity to recognize that two things/states can not be the case together. With this standard propositional logic was in place. Any further developments might proceed from there, but have to use that core as point of departure.

§5.3 *Consistency* was commonly – before the advent of paraconsistency – seen as a precondition for anything to count as a theory contender. A theory leading us into an antinomy is usually rejected. Even if paraconsistency (at least in the form of dialetheism) allows for some contradictions being true not just any contradiction in one's logical theory are acceptable.

We have to distinguish here between a logical theory being inconsistent and a formal system allowing for inconsistency. Consistency works as a constraint in paraconsistency as strong arguments are needed to overrule that requirement (i.e. the presence of the Law of Non-Contradiction) within a formal system. One type of argument put forth by the dialetheists refers to simplicity: Some of our fundamental logical or semantic principles (like the Truth Schema or Naïve

Comprehension) lead to antinomies, but these contradictions are acceptable as true contradictions, since these principles thus keep their maximal generality and simplicity (not to mention the failures of competing theories in that area). Another type of argument by proponents of paraconsistency refers to the observational requirement: Most people will not infer from some contradiction to any statement whatsoever; *ex contradictione quodlibet* is not underwritten by most people's logical intuitions. This means that logical rules that incorporate the *quodlibet* (like Disjunctive Syllogism or Modus Ponens) have to be understood as restricted in some fashion. In terms of formal systems this might mean that Modus Ponens has to be taken as a non-universal rule (like in a Default Logic or some Adaptive Logic).

§6 In addition to the general criteria of coherence a theory of logic inasmuch as it concerns cognition has to meet the further requirement of *feasibility*. Information storage and processing in humans is constrained by the general capacities of human brains and the affordable resources of deliberation in situated action. Results of computational complexity may not be easily transferred to human cognition (as complexity measures, for instance, work with worst case measures in the long run, where in applied cases an exponentially complex computation may be feasible on the usual input or a polynomial complex computation may involves too high a polynomial degree to be feasible on even small input). Nonetheless result of computational complexity theory might provide a rough assessment which rules of inference are more feasible than others. In case of alethic modal logics of necessity a further case can be made for S5 on basis of such feasibility reasoning. Propositional logic is NP-complete (by SAT being NP-complete). A modal extension of propositional logic which does not increase complexity of computing validity is *prima facie* preferable to an extension which increases complexity of computation. As S5 allows for reduction of modalities S5 is also NP-complete. In contrast weaker modal logics like K or S4, both of which involve many more basic modalities, are in a different complexity class: PSPACE. Thus S5 is vastly more feasible. Some modal logics that contain simple relational reasoning move to complexity classes EXP and EXPSPACE. A logical theory involving rules of exponential complexity has at least to add a supplementary theory what (cognitive and computational) shortcuts may help to decrease this computational bottleneck (e.g. chunking statements or allowing for small error probabilities instead of certainty).

§7 Having a proper theory of logic, and modelling the human logic faculty thus follows roughly the same methodology that other (empirical) theories of cognition do. As there is the human language faculty, there is a human faculty of logic. As linguistic theories are revised to capture the initial state of the language faculty and its growth, also distinguishing competence and performance, so logical theories have to be revised in their attempt to capture in a formal system the core logic of the human logic faculty, and to account for a possible gap between the strength of that system and its pragmatic employment in situated deliberation and communication. Evolution might revise logic, logicians revise logical theories.

§8 This (psychological) realism about logic is a realism about *representations*: rules of logic are representations, inferences are ways to process representations, logical structure structures representations. Logical realism of this kind fits into a Representational Theory of Mind. Mathematics, on the other hand, seems to come with massive ontological commitments, which carry over to logic, once its meta-theory is cast in model theory. Again one may argue that our conceptual scheme contains some basic mathematical concepts, prominently some concept of collections like extensions of concepts, sets or heaps. Completely different accounts might be given of these: like a Fregean theory of value ranges for extensions, some mereology (extensional [with Lesniewski] or intensional [with Simons]) for heaps, and some set theory (ranging from finite set theory to ZFC variants, to variants with sets and proper classes like NBG or MK, and many other versions like KP, NF and what not). In the vain of the discussion above one may ask which are the principles our ordinary concept of collection relies upon. Controversial, but supposedly obvious, candidates are Naïve Comprehension, Frege's Basic Law V, Existence of General Sums ...
One may doubt, however, whether evidence for one of the complete systems can be put forth. And

one may now ask oneself how we have to take the ontological commitments that come with these systems. Therefore we turn to consider ontological anti-realism. Ontological anti-realism in mathematics turns out to be compatible with (psychological) realism about logic and basic mathematical concepts.

Any form of anti-realism in mathematics – just as any form of realism in mathematics – has to account for

(1) the meaning of mathematical language

(2) the *a priori* nature of mathematics

(3) the applicability of mathematics to reality in the empirical sciences.

Supposedly (1) poses the greater challenge for the anti-realist, than for the mathematical realist, whose challenge is (3).

§9 On an anti-realist view mathematics is *objective* by being true only by force of the conventions laid down. Mathematical truths are derivable, and nothing but derivable. An anti-realist in mathematics *identifies* truth in mathematics with derivability. Truth in pure mathematics is 'true in the story of pure mathematics', coinciding with being derivable from the axioms of pure mathematics by the rules of pure mathematics and logic alone. Thus mathematics is also *a priori* and analytic. This answers question (2) above. Difficult proofs may enlarge our knowledge and deepen our understanding of the impact of the conventions, so even analytic sentences can be subjectively surprising and be a gain in explicit knowledge.

Pure mathematical talk has *meaning* by the conventions of pure mathematics (i.e. rules of usage and recursive truth conditions). This is half of the answer to question (1) above.

This conception of mathematical truth stands in no conflict with Gödel's Incompleteness Theorems, as one may observe (following Wittgenstein) that the reasoning establishing the truth of the Gödel sentence takes place in *another* formal system than that in question, as well as claim (following Dummett) that the non-coincidence of truth and provability in some system only shows that our intuitive resources of reasoning – as employed in the meta-reasoning – are not completely formalized, and so the respective system may be extended indefinitely, or one may even (following Priest) derive the Gödel sentence for a paraconsistent system in that very system.

§10 The ontology of pure mathematics (i.e. pure set theory) supports this objective quality by providing a picture of independently existing entities warranting and corresponding to the objective mathematical truths. This realm is a *fiction* accompanying the conventions of mathematics. To answer question (1) completely with respect to reference of expressions in pure mathematics one thus adds: we are presented a story/a picture of a realm of entities which serve as substitute referents for expressions in pure mathematics the way fictional characters serve as substitute referents for their names, which means properly speaking they *do not refer at all*, but are 'mere' representations. Pure mathematics tells a story, but not a story *about* something, neither about the 'forms' of Platonism nor the 'non-existing objects' of Noneism.

Pure mathematics, however, is distinguished from other fictions (other arbitrary conventions) by its applicability in the sciences and everyday life. One may account for this – and so answer question (3) above – as follows:

I. Pure mathematics (i.e. ultimately pure set theory) consists of a linguistic structure containing both expressions (supposedly) referring to single entities as well as those (supposedly) referring to relations and properties.

II. Parts of reality (e.g. countable objects and their measurable properties) provide *a partial model* of this mathematical structure. A model in the not set theoretic sense: these parts of reality can be linked to mathematical expressions (e.g. in case of measurable extents of qualities) *and* the derivable consequences with respect to them (derived using the purely mathematical structure) *apply again* to parts of reality.

III. This homomorphism (in the non-technical sense of the consequences of the picture being a picture of the consequences) invites us to assume that those parts of the purely mathematical language we have not fixed to some part of reality may nonetheless be understood as having a model, because we believe that we just have not observed their counterparts yet, or we treat their counterparts like theoretical entities in the sciences, or we just don't care about these counterparts as long as the homomorphism stays stable on the observed counterparts. One may take these parts of pure mathematics as useful supplementary fictions. They are 'supplementary' as there is nothing in the partial model relating to them.

The fictionalist with this relates *the whole of mathematics* to reality by anchoring pure mathematics in a partial model of it. In this view there is some truths in a realist picture which sees mathematics corresponding to structures of reality. In this view non-applied (purely pure) mathematics can be tolerated by its service to applied mathematics. It shouldn't be taken as exploring a self-sustaining mathematical reality. We can even speculate that our conceptual scheme rather contains a mereological concept of collection, which is taken up and extended (e.g. by postulating the existence of an empty set) by set theory. In that case the conceptual support for our immediate judgement about set theory and its principles may *rest in (intensional) mereology*.

§11 Pure mathematical talk fulfils, thus, another function than scientific talk. Scientific sentences are true or false, including those containing mathematical elements, as these elements are linked to procedures of establishing the truth value of a sentence (e.g. by the use of measuring paradigms). Scientific language aims at the facts in describing reality. Pure mathematics, in contrast, can be seen as either 'true to the story of mathematics' (by being derivable given the axioms) or as just *telling and establishing* the background story to applied mathematics (being objective by its intersubjectively shareable character of conventionality). The language game of pure mathematics doesn't concern the description or denotation of mathematical facts.

An overall fictionalist account of pure mathematics claims that even pure mathematical sentences talking about finite cardinals do not refer to mathematical facts, not just those purely pure mathematical sentences dealing with the remote regions of *Cantor's Paradise*. Scientific or everyday assertions containing numerals (e.g. “There are 3 apples on these 2 tables”) possess truth conditions or are linked to procedures of justification that do not involve *numbers*, even finite cardinals. Procedures and rules covering counting, measuring or employing a ruler and a pair of compass serve as bridge principles relating empirical sentences and observations to pure mathematics.

Thus the fictionalism outlined differs from anti-realistic constructivism, which at least maintains those parts of pure mathematics which have been constructed. The fictionalist doubts the use of the constructivist's further assumption that by carrying out steps of reasoning inside mathematics we have *supplied further entities*.

Besides fictionalist anti-realism there are more well-known forms of non-standard treatments of mathematics, many of which have a constructivist streak, often including their treatment of logic. One may, for instance, target the understanding of universal quantification. Contrary to Cantor's Domain Principle, which assumes the existence of a domain of values for the variables quantified over, one may understand universal quantification like a conditional (substitutional) claim: once a value (or a term) is provided the universally quantified sentence holds of it, how many whatsoever these values (or terms) may be. One need not even take the domain they are supposedly are collected from to be completely given: it might be expanding or be otherwise elusive, what counts is only the conditional claim on any values (or terms) provided. This resembles using schemata with schematic expressions instead of variables and quantifiers. Such an account trades in a non-standard use of quantification (in mathematics) for ridding itself of a fictionalist account. A fictionalist account of pure mathematics need not involve a revisionist understanding of quantification, not more than any account of fiction, as fiction in general also employs quantifiers (be it counting fairies or numbers). This fictionalist anti-realism does not 'revise' standard mathematics (or ZFC) in any sense, e.g. by changing its underlying logic. The anti-realism does not pertain to logic, but to ontology.

Identifying something as pure mathematics serves as a rigidly syntactical *indication of scope* which puts the assertive force of the involved declarative sentences into brackets. The inscription “a novel” on the title page or cover of a book informs us that we confront a work of fiction, we put what is said into the brackets of a story – almost the same applies to “a treatise in set theory”.

§12 In as much as reality only provides a partial model of mathematics reality cannot distinguish between those mathematical structures which are equivalent with respect to the descriptions and the projections covering the partial model. Therefore more than one mathematical structure can be applicable to reality, and thus be useful, and in this sense be justified.

This will be so for set theoretical differences (say in large cardinals) way beyond any direct relation to applied mathematics. Postulates of the existence of large cardinals may either be rejected as superfluous or may even be endorsed as equipping the complete mathematical structure with valuable structural properties like symmetry or non-arbitrariness. This may also be the case for the distinction between mathematical structures differing in the *cardinality* of the number classes involved (i.e. those being finite, enumerable or more than enumerable). Reality may not be – or supposedly is not, according to quantum mechanics – continuous, not even dense. Not just the rational and real numbers may be too much – even large *finite* cardinals may have no application to reality.

So, even if there is no conventionalism with respect to logic in as much as our logic faculty is concerned, there is plenty of room for *conventionalism in mathematics*. Quantifiers (of some sort) and thus some powers of counting objects are part of our logic faculty, set theory (say in the form of ZFC) almost certainly isn't. Carnap's Principle of Tolerance applies here, as well as meta-theoretical criteria of theory choice in pure mathematics (like symmetry or ease of computation).

§13 Insofar as pure mathematics (set theory) serves only as the background for applied mathematics and carries no ontological commitment by itself, we needn't be as concerned about set theoretical paradoxes and foundational problems as mathematical realists are. A story may contain unsolved puzzles or even confusions – they do not matter as long as they do not affect those parts of the story relevant to us. A novel, for example, may contain errors and confusions concerning the economy of a society depicted, which nonetheless may be irrelevant to its main plot (of character development or crime detection). In that vain foundational issues in set theory lose a lot of their interest to the mathematical pragmatist, quite contrary to the semantic and logical paradoxes, which highlight either an insufficient re-construction of our logic faculty or even inbuilt conceptual mismatch.

The focus on applied mathematics as crucial explains the lacking interest of the working mathematician in pure mathematics. If there were more than lip service paying real mathematical realists there should be much more concern about the problems of set theoretical foundations (like the status of the universe *V* or the existence of {}). Working mathematicians by their pragmatism embody an anti-realistic attitude to mathematics.

§14 Psychological realism with respect to logic and pragmatism with respect to mathematics are compatible, as the logical realist stops at the existence axioms of pure mathematics (especially the Axiom of Infinity). Even a dose of *logicism* may be compatible with anti-realism in mathematics: it may be so that our logic faculty (i.e. our conceptual scheme with respect to logical concepts) allows for the derivation of some advanced mathematical concepts and structures. Realism with respect to logic meant that we *have* this one very logic, it does not mean that all concepts employed in that faculty have objective reality in application. As often with human cognition they only have to be good enough in our dealings with reality. Thus the concepts of our logic faculty may invite and sustain some elaborate fictions of pure set theory that underwrite pure mathematics. Again one has to separate immediately applicable mathematical talk of entities, structures and consequences from scaffolding. Realism with respect to logic rests in the fact that we possess a logic faculty that is the way it *is*. Anti-realism with respect to mathematics rests in the belief that there aren't neither pure sets nor numbers of any kind. There are no facts of the matter to be discovered about them. There are matters of fact concerning our logic faculty. Our best theory of logic systematizes them.

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