Universality and the Universe in ZFC

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> Any mathematical consideration must be founded on the notion of 'allness' or 'quantification' as a basic category of logic which cannot be subject to further analysis whatsoever.
>
> (Ernst Zermelo)

> What surpasses all that is finite and transfinite is no "Genus"; it is the single, completely individual unity in which everything is included, which includes the "Absolute", incomprehensible to the human understanding.
>
> (Georg Cantor)

Standard symbols are used.

$\wp(x)$	is the powerset of *x*.
$\|x\|$	is the cardinality of the set *x*.
$\Rightarrow$	is used in rule statements

Greek letters φ, ψ … are used as schematic for unspecified formula or predicates (open formulas) of a given language. Γ, Σ, Π are mostly used for sets of formulas.

CAPITALIZATION is used to signal that we now talk about a concept.

Single quotation marks are used in quotes and as 'scare quotes' of established terms. Double quotations marks quote an expression. Names of famous theorems are *italicized*.

M	is the set of urelements
U	is the universal set
V	is the complete iterative hierarchy, V_α a rank in the hierarchy

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Introduction

This essay discusses the fate of universality and a universal set in several set theories. It presupposes a general background in logic and general knowledge of set theoretic basics. Even basic points are repeated if the context of discussion profits from a short reminder, but no systematic survey of the different systems is attempted. The book aims at a *philosophical* study of ontological and conceptual questions around set theory. A *formal* exposition of some consistent set theories with a universal set and related theorems can be found in (Forster 1992) and the sources mentioned there

Set theories *are* ontologies. They posit entities and claim that these exhibit some essential properties laid down in the set theoretical axioms. Like Zermelo (in the opening paragraphs of Zermelo 1908) Fraenkel, in his early introduction to set theory (1919/1928[2]) explicitly outlines this *axiomatic approach*:

> According to the essence of this method we refrain to *define* the concept of set or to analyze it, we rather start with some axioms in which the concept of set like the relation 'to be contained as an element' occurs, and in which the existence of some sets is postulated. The concept of set is implicitly established by the totality of these axioms.

Collecting these postulated entities poses the problem of universality. Is the *collection* of the set theoretical entities itself a set theoretical entity? What does it mean if it is, and what does it mean if it is not? To answer these questions involves developing a theory of the *universal set*. For a start we may define the universal set as $U = \{x \mid x = x\}$. As set theories extend first order logic with identity (**FOL**) or some variant of it (in a non-standard logic) they contain the axiom: $(\forall x)(x = x)$. U thus comprises the *whole domain* of the language. Tautologically whatever exists exists. So, supposedly, *there are* all existents. Why not continue: So *there is* the totality of these existents? Why shouldn't they be collectible? After all, set theoretical quantification runs over *all sets*, doesn't it? If, however, that totality was an additional entity we could collect another totality including it – and so forth, it seems. Unless this totality possessed a nature *sui generis*, setting it apart from ordinary things and sets.

Some of the set theories are pure set theories: their domain consists entirely of sets, all variables range over sets. Some of the theories contain proper classes in addition to sets (improper classes). Some theories contain additional 'urelements' (i.e. objects which are neither sets nor classes, but something to be collected into sets or classes). Some other theories use numbers as basic entities, not reduced to set theoretic construction. The discussion here will mention these differences, but will not use a neutral formalization, which applies to all theories; this would require using one type of variables and sortal predicates like "set", "ordinal" etc. to restrict quantification to the appropriate type, e.g. "$(\forall x,y)(Set(x) \wedge Set(y) \supset \ldots$". The drawback of this formalization would be its contrast to the respective textbooks and articles. Additionally, it would be very cumbersome, e.g. having all the sethood statements in pure set theories like **Z**, the axiomatization of which would include now a new axiom "$(\forall x)Set(x)$", which had to be used all over to get rid of the sethood requirements in the antecedents of statements. Thus, when discussing mixed type systems sortal predicates may be used, but not with pure set theories, and not with systems which only distinguish sets from (proper) classes; in the latter case lower case variables refer to sets, upper case variables to (proper) classes.

The existence of urelements is important for the broader ontological picture, but the presence or absence of a basic set of urelements does not change the treatment of universality in many set theories (like **ZFC**). One may ask oneself what sort of things might be chosen as urelements. If physical entities are chosen, there are – in the light of our best physical theories – only finitely many of them, which can be collected into a set of urelements. Physical objects may have their own principles of composition (like mereology). Their presence does not influence the question whether all sets can be collected into a universal set. Once sets are present, there seems to be no need for further elementary logical (abstract) entities like numbers. A collection of urelements that matches the sets in cardinality seems highly dubious, as one may suppose every urelements to have a singleton and any two of them to be elements of their pair set – etc. In some cases (like *Specker's Theorem*) we have to talk about urelements.

Several issues related to set theory will not be discussed here: We are not much concerned with the *epistemology* of mathematics in general or set theory in particular. With respect to epistemology all the theories discussed here are *prima facie* in the same boat. We may, however, raise some questions concerning whether understanding universality raises *additional* epistemological problems. We start with the ordinary working assumption that we have some concepts and ideas of sets and numbers and set theory tries to systematize them. Therefore, we will not be concerned with the general issue of abstractness or 'Platonism' either. Again, *prima facie*, all the theories discussed here are in the same boat. We may raise some questions concerning whether some ways of understanding universality or the set theoretic universe raise *additional* ontological problems.

For the set theoretical anti-realist our study is just a case study in formal ontology and its models. Nonetheless some such ontologies might be more useful than others even if all are – strictly speaking – false, as there are no sets whatsoever. Even if there are no sets some set ontologies may be more helpful fictions than others. They help in systematizing mathematics, which again, even if without subject matter itself, helps as part of science in describing reality. Even Russell held at times that sets are just a manner of speaking, but not part of the furniture of the world (cf. Russell 1914). Nonetheless as they correspond, for Russell, to the fundamental 'propositional functions' talk about them is neither arbitrary nor idle. One theory is singled out as capturing or founding *our* mathematics.[1]

For the set theoretical realist one set theory might be better in capturing set theoretical reality than another. Either one has to assume U or one has to assume that U does not exist. Our study then is one attempt to ascertain which option we have to take. Even if one endorses 'plenitudinous Platonism' (the thesis that *all* consistent mathematical theories correspond to some part of the realm of abstract entities[2]) the issue of U is not idle. The different set theories might then be taken to deal with different areas of abstract entities. In one area there might be something like U in the other area not. Nonetheless, one may argue that one of the areas has more right to be considered as making up *sets* as we have an intuitive notion of SET, which may be explicated better in one theory than in another. Even if there are several areas of abstract entities which are set-like one area may be the *intended standard model* corresponding to our concept of SET. In this perspective our study is concerned with the *conceptual issue* of analysing our concept of sets. Comparing the different theories and weighing the advantages and disadvantages of incorporating U into a set theory (i.e. the gain and the strain of related theorems/facts in relation to our intuitive understanding of sets) we may come to a result whether our intuitive concept SET involves the assumption of a universal set or rather some other picture. Even if set theory was not *of* sets in the referential sense (as there might be no sets at all) set theory would be *of* sets in the intensional sense of setting out our conceptions of sets. And our question here is whether the universal

[1] Ironically Russell demanded at that time, of course inspired by the antinomy of the set of non-self-membered sets, that saying either that a set is a self-member or that it is not should be *meaningless* (not just false). A requirement which excludes the Axiom of Foundation. Naturally Russell thought in terms of his theory of types, which by definition fulfils this requirement, but, nevertheless, was laying down *conceptual constraints* on a feasible concept of set.

[2] Cf. Balaguer 1998. The view that consistency proves sufficient to take a mathematical theory seriously has many variants. These include Hilbert's formalism and fictionalism, a theory of course denying the existence of abstract entities. Included is as well Cantor's 'theological Platonism', which has all consistent transfinite entities existing as ideas in God's mind.

set crucially belongs to these conceptions, or whether it is an idea at the periphery of these conceptions, only to be rejected on second thoughts about its consequences.

ZF asserts that some collections we have naively thought of as *sets* (the set of ordinals, cardinals, the universal set) are not sets – i.e. they do not exist for **ZF** *itself*. Most surprising is this claim for U, as {x| x = x} seems so natural. That our untutored intuitions have to be partly corrected at the foundation of science, however, occurs not just in **ZF** but – one may well argue – at the foundations of physics (e.g. with our untutored intuitions about the locality of particles or the properties of time) or in biology (e.g. with the changeability of some organisms' essence/species). Thus the mere correction of our prior, untutored understanding of a basic concept does not establish *in itself* that **ZF** goes wrong. The argument has to concern whether this is the best option, what repercussions this step has, and whether the resulting concept of sets provides a more coherent (unified and comprehensive) understanding of sets.

The antinomies of semantics and set theory have to be treated somehow to provide a coherent systematic account of the notions involved. The same applies to the presupposed concept UNIVERSALITY in set theory. One can well do in large parts without treating these problems. Many a textbook works with informal set theory. They miss then, however, a comprehensive account of sets. What their success – *inter alia* (compare similar arguments in semantics) – shows is that the problems occur placed within an otherwise viable world view or viable procedures in semantics or set theory, say some version of semantic realism or of constructive representationalism. I, therefore, neglect theories that argue from antinomies and universality to some form of mysticism, ineffability, anti-representationalism, or what not.[3]

[3] Patterson (2008) extends his anti-representational program to mathematics. Postmodern authors endorse Wittgenstein's *Tractarian* mysticism about ineffability.

The following issues set the theme for much of the discussion here:

1. How can one avoid slipping into a 'theory' that universality is ineffable?
2. Are there different aspects to universality in set theory, which stand in conflict to each other?
3. What aspects of universality are embedded within our concept SET?

and most importantly:

4. How far can axiomatic ontology take us in postulating our way out of the problems around universality?

The treatment of universality in standard set theory **ZF** raises issue (2): universal comprehension and universal collection (into a domain of all sets) seem to be incompatible. The spectre of ineffability, issue (1), raises its head. We have to ask whether **ZF(C)** has a claim to be 'natural' or 'intuitive' for us, issue (3). **ZF** also articulates one picture of universality: the iterative hierarchy. Articulating this picture raises issues (1), (3) and (4).

Although this is a systematic study (i.e. not an historical investigation into the development of several set theories) sometimes it may be illuminating to mention and consider side-remarks made by their foundational authors. In these remarks one can at times discern the broader ontological picture the author works with.[4]

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[4] As this is no detailed historical study, I often omit giving the detailed source of a side-remark, but refer the interested reader to the comprehensive studies by Ebbinghaus (2007) on Zermelo, Dauben (1979) on Cantor, as well as (Lavine 1998). Some remarks have entered the set theoretical folklore and can be found in many introductory books (e.g. Potter 2004) or (Deiser 2010), which contains many quotes of the founding fathers, following the development of set theory.

Universality in ZFC

ZFC has become in its first order axiomatization the accepted set theoretic standard. We take, as usually done, as **Z** the system containing the Axioms of Extensionality, Pairing, Powerset, Sums, Separation and Infinity. **ZF** adds Foundation and Replacement, like Separation an axiom schema. **ZFC** adds the Axiom of Choice.[5] **ZFC+GCH** adds the Generalized Continuum Hypothesis to **ZFC** – and so on for stronger axioms.

The antinomies (like *Russell's Paradox*[6]) are often taken as showing that Naïve Comprehension

(NC_1) $\quad (\exists y)(\forall x)(x \in y \equiv \varphi(x))$

(NC_2) $\quad (\forall F)(\exists y)(\forall x)(x \in y \equiv F(x))$

is wrong. The assumption that every concept/property[7] has an extension, which is a set, is considered rejected. The first order axiom schema (NC_1) or the second order axiom (NC_2) are sometimes called 'naïve set theory'. They were by no means present in all approaches to set theory introduced in the 19th century. Cantor's original set theory was concerned with combinatorial multiplicities. At times, though, he considered sets as 'united by a rule',

5 Historically this is misleading as Zermelo included the Axiom of Choice in his system, where he used it to prove well-ordering (in 1908). He also has an extra axiom for the empty set, ∅, but as in **FOL** the domain cannot be empty, one does not need this axiom, but gets ∅ by separation. In the 1920s Fraenkel and von Neumann and Skolem added Replacement. Zermelo's original system did not contain Foundation, but his system of 1930 does. His 1930 system **ZF'** leaves out the Axiom of Infinity as he then considered it to be an extra-logical existence assumption. Zermelo's formulation was not confined to **FOL**, but Skolem's clarification of 'definite' property as used in an instance of Separation led to first order **ZFC**. Cantor already stated and used both the Axiom of Choice and Replacement.

6 'Antinomy' will be used for a contradiction provable given some theory and its logic. A 'paradox' is just a theorem contrary to our expectations and prejudices. Already Zermelo stressed the importance of this distinction, as otherwise one sees the likes of too many antinomies where there are only paradoxes. Unfortunately, usage is not so clear nowadays. By the way: The antinomy unfortunately called 'Russell's Paradox' was discovered some years earlier by Zermelo. It leads back – as many antinomies – to negative self-application of a property/predicate, the idea behind the canonical proof of *Cantor's Theorem*, which served as the context of discovering 'Russell's Paradox'.

7 In the context of this essay I take "concept" and "property" to be synonymous within set theories, as is usually done. In (natural language) semantics concepts may be said to refer to properties, which are often not taken as sets. Set theoretic ontology is less fine grained. A distinction is made between formulas expressing a concept/property and the concept/property. CAPITALIZATION is used to signal a concept/property. Reflecting *on* set theory and its relation to our cognition concepts (like the concept SET) are taken in their usual sense as cognitive, and whether they are captured and explicated by a theory (say, of 'sets') is the matter of debate.

which sounds like Comprehension. Comprehension was certainly present in the logicist approach to set theory of Frege and Russell.

Now, take a version of Comprehension: the Russell Set, defined as R = {x | x∉x}, and taking 'x∉x' as the open formula φ(x) or the property F yields the famous antinomy: R∈R ∧ R∉R. The defining property of NOT BEING A SELF-MEMBER seems to violate the constitutive assumption behind Naïve Comprehension by not having an extension, on pains of inconsistency.

There is another reading of *Russell's Paradox*, however. Proceeding to Zermelo's *Aussonderungsaxiom* (Axiom of Separation)[8] or not-naïve Comprehension scheme (of set theory **Z**)

(AS) $(\forall x)(\exists y)(\forall w)(w \in y \equiv w \in x \wedge \varphi(x))$

the property NOT BEING A SELF-MEMBER can be used to derive:

(NU) $\neg(\exists x)(\forall y)(y \in x)$

the denial of a universal set.[9] What *Russell's Paradox* shows on this reading is that the assumption of the existence of a universal set is illicit. *Cantor's Theorem* establishes that the powerset $\wp(x)$ of a set *x* has a larger cardinality than *x*. Cantor's classical proof refutes the supposition of a bijection *f* between *x* and $\wp(x)$ by considering the subset $\{x \mid x \notin f(x)\}$. If *x* is the universal set this naturally introduces the Russell Set (being an element and a subset of the universal set). The idea of a universal set thus stands in tension to a core ingredient of the concept SET: that every set has subsets, which should be collectible. "⊆" is as central to set theory as "∈": one of them provides a sufficient foundation:

(D⊆1) $x \subseteq y \stackrel{\text{def}}{=} (\forall z)(z \in x \supset z \in y)$

(D∈1) $x \in y \stackrel{\text{def}}{=} \{x\} \subseteq y$

8 To be precise: It is a schema in the wff φ. Any set can be separated by *this* axiom schema which corresponds to a wff in the language of the theory. The constructible universe L (used in Gödel's relative consistency proof for the Axiom of Choice and the Continuum Hypothesis) consists only of such sets, which requires restricting the powerset operation to constructible subsets.

9 *Proof* (Outline). Assume U exists. Take U as the base set *x* in (AS). The first conjunct on the right side of the biconditional can then be eliminated, being logically true. One arrives at the form of (NC_1) and the usual reasoning to the Russell Paradox goes through. Reject the existence assumption concerning U by arriving at the contradiction. ■ This proof can already be carried out in a weak subsystem of **Z**, like Kripke-Platek set theory **KP**. Membership can hardly be indeterminate for a set theoretic realist. Even if this had some plausibility for some sets, with respect to U something is in the universe or is not. Avoiding the *Russell Paradox* by banning R from U leaves us with the mystery where to put R then, or with the option that some collections cannot be sets, which leads to a set/non-sets distinction. Indeterminate membership plays no role here; theories without *tertium non datur* will not be considered here, but giving up *tertium non datur* may mean rejecting *both* R∈R and R∉R.

The Powerset Axiom focussing on “$\subseteq$”, therefore, deserves a special role in any set theory, as Comprehension and/or Separation focus on “$\in$”. That $U \in U$ may seem less unnatural than $\wp(U) \subseteq U$ and $\wp(U) \in U$.[10]

Comprehension is fine as long as we restrict the domain of objects to be comprehended. If we assume that there is no universal set or domain even Naïve Comprehension need not lead to the antinomies, as one cannot take for granted that R (or a similar cause of trouble) belongs to the objects (sets) to be comprehended. (AS) provides the safe formalization of this idea. The property NOT BEING A SELF-MEMBER can be taken as having an extension now that (AS) has been adopted. Any property has an extension *relative* to a base set. And if *a* is the base set for an instance of (AS) with ‘$x \notin x$’, the extension of the subset corresponding to NOT BEING A SELF-MEMBER relative to *a* is *a* itself (as by the Foundation Axiom no set is a member of itself, so that all members of *a* satisfy the condition $x \notin x$).

The discovery behind the set theoretical antinomies then consists not in a claim about properties

(NNC) Not every property has an extension.

but in a claim about universality

(NU’) There is no universal set.

Both claims are ontologically substantial and surprising. Hilbert, for instance, thought that conception formation was in trouble, as the idea that being able to determine whether something falls under a concept does not suffice for the concept’s existence.

The argument against U works with Separation. Using (NC) leads to the antinomy. One reading of the antinomic argument can also be that it uses the assumption that the Russell Set *R* is part of ‘all’ objects (i.e. *within the range* of “$\forall$”). The range of “$\forall$” on pains of contradiction thus cannot be universal, *R* lying outside of it. Thus there is no unrestricted quantification over all collections. If “$\forall$” ranges over all sets, *R* cannot be a set after all. The collection of non-self-membered sets turns out *to be* the range of “$\forall$” in **Z** because of the Axiom of Foundation (i.e. turns out to be the iterative hierarchy V itself)! In this reading of the antinomic argument again *a set* of all sets is excluded. The reasoning poses two problems we come back to again and again: (i) (NC) still allows building the forbidden collections U

[10] Even the problem with Frege’s ‘basic law’ (V) goes back to this, since Frege at the same time defines extensions as objects (i.e. first order entities) and puts them in basic law (V) in correspondence to courses of values (predication) of concepts (i.e. second order entities), by *Cantor’s Theorem* there have to be more extensions of concepts (namely sets of objects) than objects (cf. Boolos 1998, pp.135-54). Because of the complete absence of a Powerset Axiom we do not consider set theories like **KP** (Kripke/Platek set theory) in detail.

and *R*, and (ii) the reasoning invites our naïve bewilderment *where* some collection is 'to be' when outside of the range of "$\forall$".

The naturalness of the idea of universality or a universal set may be related to the *Calculus of Classes* (cf. e.g. Hilbert/Ackermann 1928, Chap. 2).[11] Textbooks unhesitatingly speak of a 'universal class' here. The *Calculus of Classes* systematizes our reasoning with respect to 'classes' of arbitrary objects by defining cuts, unions etc. The complement of such a 'class' a is an absolute complement $\bar{a}$, such that $a \cup \bar{a}$ is the 'universal class'. The crucial point is that these 'classes' of the *Calculus of Classes* only contain individuals of the considered domain. There are no 'classes of classes'. The 'universal class' is just the domain considered. The 'classes' of the *Calculus of Classes* are neither sets nor classes. They obey some axioms (like Extensionality), but others (like Powerset) do not apply here. The concept SET exhibits much more complexity than the concept COLLECTION OF INDIVIDUALS!

Given the logical apparatus of **Z** we can even derive: $U = \{x \mid x = x\} = \emptyset$, *even though* we have: $(\forall x)(x = x)$![12]

There are several reasons why there is no universal set in **ZF**:

1. There is no U because this contradicts *Cantor's Theorem* (i.e. because of the Axiom of Powerset). For U we should have $\wp(U) \subset U$, but this contradicts *Cantor's Theorem* (as, trivially, a subset has at most the cardinality of the superset). [By the way: Hilbert had a similar argument working with self-mappings of functions of numbers.]
2. There is no U because this contradicts the Axiom of Foundation. For U we should have $U \in U$ against Foundation.
3. There is no U by the Axiom of Separation, as shown above.
4. As, because of further antinomies, there cannot be a set of all cardinal numbers or of all ordinal numbers – as was already clear to Cantor – there can be no U, which had to contain these sets as separable subsets.

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[11] In the following paragraph "class" is scare-quoted to make clear that these collections are not proper classes, but collections of individuals.

[12] *Proof* (Outline). If one allows for *definition by abstraction* in a pure set theory (i.e. without *atoms*, which are not sets) one has to use a scheme like the following:

$$\{x \mid \varphi(x)\} = y \equiv ((\forall x)(x \in y \equiv \varphi(x) \wedge (\exists w)(w = y)) \vee (y = \emptyset \wedge \neg(\exists w)(\forall x)(x \in w \equiv \varphi(x)))$$

Now, for an instance of this scheme with $U = \{x \mid x = x\}$, assume $U \neq \emptyset$, then the second disjunct on the right hand side is false. Therefore the first disjunct has to be true. This leads to contradiction again, by the proof for (NU). Thus the assumption has to be rejected. [In a set theory with atoms the second conjunct in the first disjunct has to be the meta-linguistic assumption that *y* is a set, cf. Suppes 1960, p.34.]

5. There is no U by the Axiom of Pairing in combination with Foundation as {U} could be built by Pairing (i.e. U and U again gives {U,U}={U}), but {U}∈U contradicts Foundation as {U} does not have an element that does not share an element with it (as U∈U).

The absence of a universal set yields more consequences in **Z, ZF** and **ZFC**. In **Z, ZF** and **ZFC** absolute complements are missing: since subsets are separated relative to a base set the complement to a set *x* is not the collection of all things not in *x*, but only the collection of those things in the base set which are not in *x*. This follows the spirit of Separation, but violates, supposedly, our intuition as to complements. Just as Comprehension is restricted in **Z** so is complement building. There cannot be absolute complements as the absolute complement to ∅ had to be U.

As **ZF** and **ZFC** are naturally understood by the iterative hierarchy [cf. below] their definition of number cannot be Frege's. Frege used a flat universe and defined a cardinal number as the equivalence class of sets with the same equinumerosity – or a representative of that equivalence class.[13] Frege defined equinumerosity by means of bijective functions. This cannot be done in **ZF** as, for instance, there are singletons of any rank in the hierarchy, so the supposed set representing 1 had to contain elements from *any* rank, but this is impossible for a set (contradicting the Reflection Principle): Sets have a minimal rank, the rank at which all their elements are present. A collection of sets of arbitrary high rank cannot be a set, and this cannot be or represent a number.

The idea that there is no universal set seems to go against our logical intuitions as we have developed them working with quantificational logics: There is always a *domain of all objects* to be quantified over.

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[13] In fact, in Frege's *consistent* system behind the *Grundlagen der Arithmetik* the concept BEING-IDENTICAL-TO-ONESELF should have an extension, and thus a number: the number of *all* things! The system can, however, not tell us what number this is (cf. Boolos 1987).

The Universe V (of Z, ZF, ZFC)

What then can be the semantics of **Z**? How are its quantifiers to be understood? Although there is no universal set, there is *universal quantification* in **Z**. The axioms witness this. The Axiom of Separation, for instance, says of *all sets* that for any condition the corresponding subset exists. In terms of the iterative hierarchy [cf. below] the axiom talks about sets *of any rank*.

One issue should be made clear at the very beginning: The metaphors usually employed when setting out 'the construction' of some sets, say of the transfinite ordinals, should not be taken literally as involving some *temporal* procession of *arriving* at ever larger ordinals, ranks or cardinalities. As sets are abstract entities they do not depend in their existence on any one – not even God – counting up to them. Sets are simple *there*. *All of them* are there. The metaphors of construction merely serve to express the structures the sets employ, and may serve, sometimes, as didactic devices how we come to understand some set on the basis of another collection of sets. Thus, that there is no highest rank in **ZFC** should not be misunderstood as the set theoretic hierarchy V being under construction. All sets are there, thus V *is there*. For this *ontological* thesis and corresponding universal quantification it is irrelevant whether we have epistemic means to distinguish that totality from any incredibly large, but not total collection/set.

Like **FOL**, which does not count its domain to be one of the objects to be quantified over, **Z** itself need not talk about its domain. A stronger meta-language may be used to model the semantics of **Z**, typically a second order logic (**SOL**) talking about proper classes, one of which may be the domain of **Z**. We come back to this later.[14] But suppose there to be such a model for **Z**. What should the domain of it be called? It certainly looks like a universal set, as it comprises all sets. Then **Z** cannot be complete, since it does not deal with all collections of objects/sets. But wasn't it supposed to be complete in its application? V has to be a collection of sets, and can be no set itself in **Z**. Zermelo (cf. 1908) recognized this and concluded from the reasoning about the Russell Set that the domain of set theory 'is not itself a set'. There seem to be totalities beyond sets then.

[14] I use "set" to talk about sets and "class" to talk about *proper classes* (so called because these classes either *are not* sets or have no *corresponding set*, both usages are common, we come back to the idea of 'correspondence' below). "Set" and "class" are thus *not* taken as synonymous here. All claims and theories referred to are adapted to this usage; formalisms/symbols are also rendered into the common format used here. Following ordinary usage equivalence sets are called "equivalence classes" although they are no classes.

The standard picture of the realm of sets accompanying **ZF** and **ZFC** is, at least nowadays, the iterative or cumulative hierarchy. It can be argued that Cantor had already a conception of sets congenial to this picture, because Cantor thought of sets as build by the iterative application of set building functions. Frege's set theoretic universe, in contrast, has to be conceived as flat (non hierarchic). The hierarchy was clearly developed by von Neumann (1929), wherefore it is sometimes called "von Neumann hierarchy". Zermelo developed a similar picture in the late 1920s. The Axiom of Foundation and the Axiom of Replacement determine this picture. Foundation expresses the idea that a set occurs at some earliest level in the hierarchy (as sets are build/defined by iteration of set building operations there is some – though possibly transfinite – number of preceding set building operations). As mentioned before, talk of 'building' sets should not be taken as a process of construction, but only as an easy way to express structural dependencies between sets *all being already there*. The Axiom of Replacement expresses the continuation of ever higher levels (e.g. by collection a transfinite sequence of iterations of applying the powerset operator into a single set).

In the *pure* version of the hierarchy the starting level (or 'rank') V_0 is $\varnothing$, then there are two ways of proceeding to higher ranks

$V_{\alpha+1} = \wp(V_\alpha)$ for successor ordinals α

$V_\delta = \cup\{V_\alpha \mid \alpha < \delta\}$ for limit ordinals δ

the set theoretic universe V can then be seen as a hierarchy where later sets depend on preceding sets (although, of course, not in a temporal manner). The hierarchy is iterative as the two hierarchy building operations are applied over and over again. The hierarchy is cumulative as the sets present at V_α are also present at all levels V_δ with $\alpha < \delta$.[15] Each set has some earliest rank of occurrence. All ranks are transitive sets (i.e. contain all members of members of members…). The strength of the operation of collecting the powerset provides the plenty of the next stage. Reflections about how strong the idea of a powerset is concern directly the issue of the Generalized Continuum Hypothesis (GCH).

The picture is slightly different in a set theory with urelements. The set of urelements M lays at the foundation of the hierarchy $V_0 = M$. The two ways of proceeding are accompanied by the requirement that for each V_α, $M \subseteq V_\alpha$. A corresponding set theory needs to distinguish sets from non-sets and is called **ZFU** or **ZFCU**.[16]

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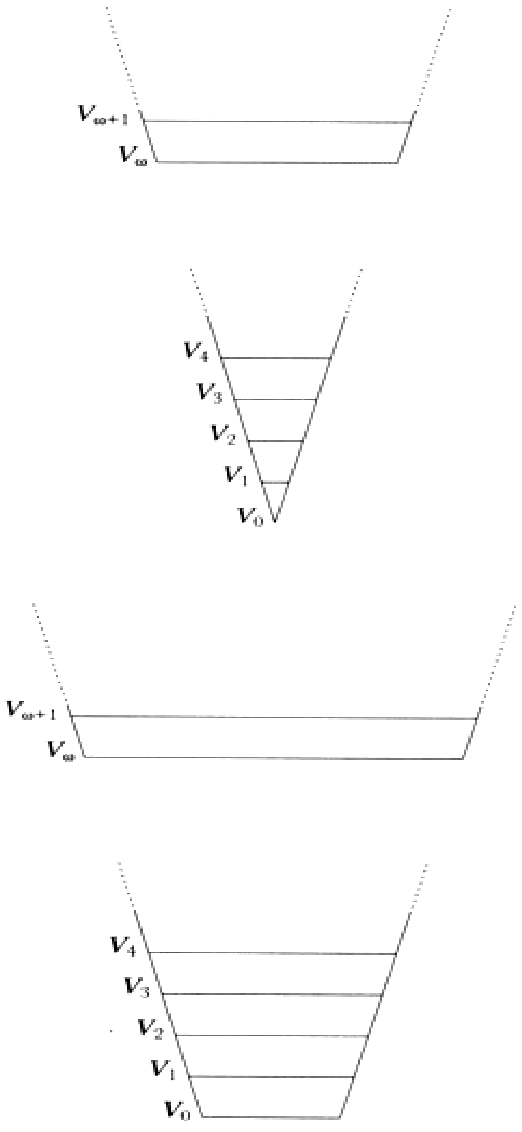
[15] Remember that $\varnothing \subseteq V_\alpha$ for any V_α as V_α is a set. Thus at $V_{\alpha+1}$ $\varnothing$ and $\{\varnothing\}$ are present and thus each stage contains all preceding stages.

[16] Usually the system is called **ZFU**, with U being the set of urelements. The name "ZFU" may thus confuse in the context of our investigation into the existence of the universal set U. Nonetheless we stick with the usual name "ZFU" as urelements and thus

Thus, one can picture V as either a pure hierarchy of **ZF**, **ZFC** (upper part in the picture) or a hierarchy based in domain of non-sets (lower part).



ZFU has a broader base than **ZF**. The dots before V_ω indicate that V_ω is the first limit level (of transfinitely many).

Z takes us with the Axiom of Infinity to V_ω, but not to arbitrary high ranks in V. We need **ZF** (i.e. Replacement) to go further. By Replacement we know that the function in *n* for $n \in \omega$ which takes as value the n-time powerset of ω has as range *a set*, since ω is a set (by the Axiom of Infinity). Therefore (by the Axiom of Union) the union of all these powersets exists as *a set*, and

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ZFU and **ZFCU** play no vital role in this book. For us it is important to distinguish the *set* of all sets U from the *class* of all sets V, so we need the name “U” in addition to “V”.

thus as a next rank in V. Now we can move in **ZF** beyond $V_{\omega+\omega}$. Note also that in this rank all other **ZF**-axioms are satisfied, while – by Foundation – the rank is not a member of itself, which establishes the independence of Replacement from the other **ZF**-axioms.[17]

Up to V_ω we find in pure set theory the *hereditarily finite sets*. They fit naturally to defining the ordinals in von Neumann's way: $n+1 \stackrel{def}{=} n \cup \{n\}$ and take $\varnothing$ as 0. Then in V_ω a transitive set of transitive sets is a number. We get: $n \in V_{n+1}$, $n \notin V_n$, $V_n \in V_{n+1}$, $V_n \subseteq V_{n+1}$. Ranks and numbers thus are $\in$-ordered. The hereditarily finite sets fulfil the axioms of **ZFC** save the Axiom of Infinity, although the Axiom of Choice and the Axiom of Replacement become unimportant here: The Axiom of Infinity is thus independent from the other **ZFC**-axioms. The finite system is sometimes called: **ZFC**$^{-\infty}$. In fact one could add an Axiom of Finiteness here:

$$\neg(\exists x)(\varnothing \neq x \wedge (\forall y)(y \in x \supset y \cup \{y\} \in x))$$

Obviously, the Axiom of Finiteness is true up to V_ω, i.e. for all hereditarily finite sets. And equally obviously V_ω (i.e. the domain of that theory) is not finite. We meet the same situation as with Quine's basic finite arithmetic [in chapter III]. Even **ZFC**$^{-\infty}$ can do what Peano Arithmetic, **PA**, does: prove theorems concerning representability and provability (e.g. Tarski's and Gödel's theorems.[18] Note that the hereditarily finite sets provide an *intended* model for **ZFC**$^{-\infty}$ (i.e. in contrast to other unintended countable models for **ZFC**). Note also – and this may be thought to be important – that Naïve Comprehension causes *no trouble* within the hereditarily finite sets. The Russell Set, for instance, does *not exist* up to V_ω as it contains all hereditarily finite sets, since they satisfy Foundation, and thus is infinite. If the set of urelements is finite as well – as one may expect in a finite physical universe – this finite consistency of Naïve Comprehension may be the background of

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17 Remember not to confuse the indices of ranks above V_ω with theses about the cardinality of the rank itself, the order type of its largest member or the index number occurring for the first time at that rank. ω+1 (i.e. {2,3,4,…1}), ω+2, ω+3 etc. are, because they are order types (i.e. relational) subsets of ω×ω, thus countable, thus sets of ordered pairs (i.e. given the usual definition of ordered pairs, sets of sets of sets of natural numbers) being subsets of $V_{\omega+2}$, members of $V_{\omega+3}$. These ranks have cardinality $\aleph_2$, $\aleph_3$ respectively and contain many, many ordinals. The set of real numbers, under the usual construction (as a set of sets of sets of natural numbers), is a subset of $V_{\omega+2}$, member of $V_{\omega+3}$, and is uncountable, whether it has a order-type (not just a simple ordering, but a well-ordering) is not obvious and is ensured only by **ZFC**, not **ZF**.

18 Cf. Fitting 2007. The Peano/Dedekind-Axioms for the successor function and induction follow easily in **Z** from the Axiom of Infinity. Taking natural numbers as von Neumann ordinals makes obvious that 0 is no successor and that the successor relation is functional. Induction follows since an inductive property is inherited by the successor relation, thus contains ω.

our intuitive support of Naïve Comprehension. Let us note this as a theorem ("y" not occurring in φ as always):

(FNC) $|\{x|\varphi(x)\}| < \aleph_0 \supset (\exists y)(\forall x)(x\in y \equiv \varphi(x))$

Let us leave **ZFC**$^{-\infty}$ behind and look at all ranks in V. With a little pretense we can say: In the iterative hierarchy *exists at some rank any proper subset of* V, i.e. (a) pretending for the moment that the non-set V has subsets and (b) speaking only about collections that can be sets (excluding a set of ordinals etc.). We can approximate Naïve Comprehension up to an arbitrary rank: y = {x | φ(x)} exists for *any* φ as long as the rank of *y* < α for some ordinal α. The set *y* exists then somewhere below α. We can say in general: If a set *x* exists *x* has some rank.[19] Existential statements are, if true, true in parts of V. The Principle of Reflection correspondingly claims that if a general sentence or a finite collection of sentences in the language of **ZFC** is set theoretically true, there is a least rank V_α which can serve as its model (with variables in the sentences bounded to rank V_α).[20] One might expect that as all specific sets mentioned in a sentence have a rank. Limit ranks ensure this structure. Once again – as with Naïve Comprehension – we seem to *approximate* talk of *all* sets! The Reflection Principle is equivalent to the Axiom of Replacement.[21] So the fully developed picture of the iterative hierarchy established by Replacement approximates universal set theoretic talk. Unfortunately, this would be too good to be true.

On the one hand we approximate universal set theoretic talk. And not just – one may claim (as Kreisel 1967 did) – set theoretic talk: Set theory can be considered to be our strongest formal system, the system to be used in the meta-theory of all other systems. Then: If some claim in some informal

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[19] *Proof* (Outline). If *x* existed without a minimal rank at which it exists, *x* would contain all ordinals as a subset, which is impossible.

[20] This does, of course, not hold for an *infinite* collection of sentences as all infinitely many instances of the schema of Replacement enforce V. The Principle of Reflection is another reason why **ZFC** cannot be finitely axiomatized: If **ZFC** could be finitely axiomatized, then it would establish – by the Principle of Reflection – a model of itself, thereby establishing its own consistency, contradicting *Gödel's Second Incompleteness Theorem*.

[21] *Proof* (Outline). The Reflection Principle entails Replacement, since if the antecedent of Replacement is true, there has to be a rank V_α modelling it; the set postulated as existing in the consequent of Replacement will be a subset of that modelling rank V_α. Replacement entails each instance of the Reflection Principle in going through the quantifiers of the finitely long compound (∀x)φ(x) taking the lowest possible rank of satisfying instances (which have to be there to make φ(x) true) and uniting them and their dependencies (by a Replacement function) into a highest most comprehensive rank, which thus models (∀x)φ(x). ■ Omitting the Replacement schema and restricting separation to formula φ with quantifiers bounded to some set provides a further weakened theory $\mathbf{Z}^-$ (also known as 'MacLane Set Theory'), which nonetheless proves sufficient for most of mathematics.

system is intuitively valid and can be captured in some formal system it has a set theoretical model. *Kreisel's Thesis* so states: Whatever is valid is valid in a set theoretical model, and if – as we may suppose – finitely many sentences were used in that piece of reasoning, it is valid at some rank V_α.

On the other hand, however, we, obviously, *shift the domain of reference* from V to some rank V_α. So, a universal statement (say, the Axiom of Pairing) does no longer talk of all sets, but only of those up to V_α. Seen in this light the Principle of Reflection resembles the *Löwenheim/Skolem-Theorem* in allowing for non-standard or *unintended* models of universally quantified set theoretic sentences. As V_α can be arbitrary high one may see this as less concerning than the countable models ensured by the *Löwenheim/Skolem-Theorem*. If V_α is a sufficiently high transfinite rank we approximate universal talk. We can also understand the possible shift of domain of reference as underlining the insight that universal set theoretic talk is bound to *strong* axioms like Replacement.

The universe V is not reached by any ladder ('construction principle') used within it. It is as strongly inaccessible by such steps as it can be. Otherwise, we only have a temporary halting point V_α. V is no number, is no set, no union or power of sets. V can only be thought as *sui generis*. How do we know this? Because otherwise it could be superseded in one of the usual ways. We thus have a *transcendental argument* concerning V's nature: it cannot be otherwise, since otherwise it wouldn't be.

Without the Axiom of Foundation or endorsing an Anti-Foundation Axiom the realm of sets is larger containing with the unfounded sets more collection like entities. Where are *these* collections collected in? U seems a good candidate for a collection of unfounded collections as $U \in U$ itself. But unfortunately, **Z** forbids U. Are unfounded collections sets? Or does our concept SET entail that sets are grounded collections? In this case we had the problem that on the one hand we had to endorse the Axiom of Foundation, but this excludes U from our set theory. If sets are abstract entities nothing seems to exclude that they contain themselves as all spatial images are inappropriate. Picturing non-well-founded sets by graphs (cf. Aczel 1988) shows easily membership bending back to its origin. Anti-foundationalist set theories contradict our concept of set, however, if set identity becomes more than identity of membership (cf. Aczel 1988, chap.4). The iterative hierarchy motivates our picture of sets as well-founded by stressing the idea of ontological structural dependence between a set and its members. In this light a set containing only itself, $x = \{x\}$, seems unnatural. U, in contrast, contains besides all other things itself. We might recognize U as a set *sui generis* and allow for U what we do not allow for other sets. Foundation would make an exception for U. But the exceptions would not end here as U, being subject to the others axioms if still a set, is exceptional – even inconsistent – with respect to *Cantor's Theorem*, for instance. Foundation certainly is built in the iterative hierarchy and V does not pose the problems with respect to Foundation that U does. According to the story of the iterative hierarchy, unfounded sets do not exist. The Axiom of Foundation follows from the set up of the cumulative hierarchy. The two conditions to proceed to higher ranks ensure the axioms of Pairing, Sums, Powerset and Infinity. Coupled with the idea of sets being extensional the structural properties of the iterative hierarchy thus *entail* the **ZF** axioms (cf. also Boolos 1989).

There are – besides the question of an Anti-Foundation Axiom – incompatible set theoretic axioms (like the Axiom of Choice *vs*. the Axiom of Determinacy[22]), which shows that there are related realms of set-like entities (sharing the basic axioms), but which cannot be consistently united. There might be a unified inconsistent realm of all these sets [cf. Chap. V]. Even the incompatibility need not show that *our* concept of set is not settled. One of the set theories may be thought to be more natural. Even a concept SET settled in its basic aspects (like set separation and powerset existence) may leave some questions unsettled. The (Generalized) Continuum Hypothesis is the best-known example. The simple Continuum Hypothesis $[\neg(\exists x)(\aleph_0 < |x| < 2^{\aleph_0})]$ is even independent of the Axiom of Choice.[23]

V has sets of arbitrary high rank. V itself does not occur in the hierarchy itself. V taken as the proper class of all ranks in V is a model of **ZF**. If V exists **ZF** is consistent, as V satisfies all its axioms. Large cardinals (strongly inaccessible cardinals beyond the reach of any set building operation by

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22 Cf. Jech 2003, pp.627-43. The Axiom of Determinacy in so-called 'Descriptive Set Theory' contradicts the Axiom of Choice, what one may take to be bad enough. It also entails some strange results for large cardinalities (like $\aleph_1$, $\aleph_2$ being measurable cardinals, but $\aleph_3$ … not being measurable). **ZFC** seems closer to our conception of sets in this regard.

23 *Proof* (Outline). Alephs are defined as infinite well-orderable cardinals. The Axiom of Choice is equivalent to the statement that any infinite cardinal is an aleph (as it implies the Well-Order Principle). Negating the Axiom of Choice (and thus the Well-Order Principle) one may endorse the simple Continuum Hypothesis *but* maintain $2^{\aleph_0} \neq \aleph_1$, since one may now deny that the Continuum can be well-ordered, whereas the *combination* with the Axiom of Choice entails $2^{\aleph_0} = \aleph_1$, since the Axiom of Choice entails that any infinite initial ordinal is an aleph. ■
Cantor proved in 1883 that there is no cardinality between the cardinality of the collection of finite ordinals ($\aleph_0$) and the cardinality of the collection of all countable *well-orderings* of ω, that cardinality thus being the next well-orderable cardinality: $\aleph_1$. Given the Continuum Hypothesis $2^{\aleph_0}$ is the cardinality of all countable *linear* orderings of ω. Given the Well-Order Principle thus $\aleph_1=2^{\aleph_0}$.
The Generalized Continuum Hypothesis (GCH) entails the Axiom of Choice: Using the first aleph GCH claims for all infinite cardinals $x = 2^{y+\aleph_0}$. $x = 2^{y+\aleph_0}$ entails *x* being an aleph, which makes *y* an aleph. The GCH thus excludes that there are cardinals in between well-orderable cardinals (i.e. in between well-orderable sets), so that *via* its respective cardinal number any set can be well-ordered. ■ (On arguments and intuitions around CH and GCH cf. Potter 2004, pp.266-82; Maddy 1988, §2.)

being uncountable, regular and greater than 2^{δ} for any preceding cardinal δ), if existing, are such models as well. For V the axioms of **ZF** are construction principles and thus trivially satisfied. For (strongly) inaccessible cardinals the important observation is that they are assumed to be just larger transitive sets. Take the least such cardinal; any function within it is of lower rank as the cardinal itself; thus, the range of the function is a set, which has this least inaccessible cardinal satisfy the Axiom of Replacement – the other axioms are obviously satisfied again (cf. Jech 2003, pp.165-67).

Having *all* subsets of a rank present at the next rank suits the Axiom of Choice: If a family of non-empty sets *x* exists at some rank V_{α}, the members *y* of that family exist already at lower ranks V_{δ} with $\delta<\alpha$, and their members *z* exist already at lower ranks V_{γ} with $\gamma<\delta$ (relative to a δ for some *y*); thus as these *z* are elements of some V_{γ} a *set w* containing one of them for each $y \in x$ exists (at the latest) at the rank V_{α} of *x*. Choice is natural in the iterative hierarchy. V rather corresponds to **ZFC**.

Once we have one of the inaccessible cardinals or the class V of all sets we have a model of ZF and could be content with respect to our theory of sets. So, should we care about their nature?

Leaving V to the side for a moment let us consider large cardinals. We have just talked about them, so we know something about their nature and we can ascribe properties to them. So, they should be the objects of some theory.

Zermelo thought of strongly inaccessible cardinals (his 'Grenzzahlen') forming themselves an unbounded sequence. This, however, implies that we quantify over them, and are again in the situation of asking over what domain now our quantifiers run. Is this collection of *Grenzzahlen* itself some *Grenzzahl*? Supposedly not to avoid antinomies of the Burali-Forti-type. Then again if we now introduce *Super-Grenzzahlen* we can start all over again with them – and once more the whole process iterates. Zermelo thought: 'This series reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, …' (1930, p. 47).

Now, this way of thinking may be innocent for a constructivist, but for a set theoretic realist the idea that sets have to come into existence is simply wrong. Placing them at some rank in the hierarchy does not mean that they come later (in time?) than the other sets. Frege's universe is anti-foundational. And for a Platonist an anti-foundationalist universe has the advantage of keeping all ideas of stepwise construction at bay. As all abstract objects are there they exhibit some ontological dependencies, but this does not require that some are before or beneath others. Impredicativity is no problem in such an anti-foundational universe. Zermelo himself rejects any spatiotemporal associations. A well-ordering 'has nothing at all to do with spatiotemporal arrangement'. He also thought the term "choice" to be problematic as one may associate (temporally) successive choices being performed, where we have only a representational/selectional correlation (cf. Ebbinghaus 2007, p.69, 135). The ranks express a structural dependency only. All ranks are there. In the same way all of that coming beyond the set theoretic ranks (i.e. any large cardinal) *is there*. Thus, there should be *a collection of it all*. Assuming a sequence of large cardinals thus does not seem to solve the problem of collecting sets, but either adds the issue of an *incomplete universe* or means that V contains them all and only our *set* theory, say **ZFC**, is *not complete yet* and has to be strengthened by further axioms.

The issue of large cardinals is independent from that of the universe of sets. If one can argue that some idea of some type or large cardinals comes from our concept of sets – say, why should $\aleph_0$ be the last inaccessible? – then these large cardinals may be thought of as stages in V above those which **ZFC** (so far) treats of. *Any* type of closure operation on preceding collections should correspond to *a set* within V. This idea resembles the content of the Reflection Principle: Any finitely specified closure condition can be modelled by some rank. Large cardinals may provide a universe and a model for **ZFC**, but they differ from classes in being collectable themselves and thus being members of the overall universe of (extended) set theory. Another argument for such additional sets stems from Scott's proof that V≠L given large cardinals, as the notions of (unrestricted) powerset and uncountability stand in conflict to V=L. The constructible universe seems unnatural, even though V=L entails the Axiom of Choice and the Generalized Continuum Hypothesis, excluding it speaks in favour of large cardinals. The constructible universe violates the idea of purely *extensional* sets inasmuch as pure extensionality should allow for sets beyond any descriptive powers. One might think that it follows the idea of Naïve Comprehension, that sets correspond to properties, but why should all objective properties correspond to formulas in the first place? Proceeding to the next rank by the full powerset operation suits the simple idea of the powerset. Curtailing the powerset to subsets which are definable leaves out sets that should *be there*.

V is the ultimate model of the universe also in the sense that constructions like 'forcing' or means of building 'inner models' start from V (cf. Arrigoni 2007; on the formal details cf. Jech 2003, pp.175-223).[24]

The iterative hierarchy does not know several classes. It might be preferable not to call V a class, but to treat of V as a very special object in its own right – an issue of axiomatic ontology. If we call V a class it is not to be thought of in the manner of **NBG** or **MK**, since there is no part of set theory which

[24] Leaving here to the side the problem that such models are non-standard or unintended, e.g. in being countable; cf. the remarks in the next chapter on limits of expressivity. One may add that inner models like L, which restrict the powerset operation, but satisfy the others axioms in their standard reading (relative to the shrunken universe), are less non-standard than models generated by forcing.

addresses it, like Comprehension and Limitation of Size address classes in **NBG** or **MK**. V is not in the range of set theoretical quantifiers. It is not in the domain. Calling V 'a class' in the context of the iterative hierarchy and **ZFC** means there exists only one class (outside of our theory of sets).[25]

V *is* the range of the quantifiers in **ZFC**. Cantor claimed that every potential infinite presupposes an actual infinite 'and cannot be thought without it' (cf. Cantor 1887). This is the Domain Principle: Speaking of and quantifying the *x* presupposes the domain of the *x*.[26]

V is a very special entity, both within the picture of the iterative hierarchy as in our meta-theory modelling our theory of sets. V has no subsets as V is no set. V is not well-ordered – even in the presence of the Well-Order Principle only *sets* are well-ordered. V is not the domain of a (replacement) function, sets are – and so on. V contains all ordinals and all cardinals, but there is neither a set of all ordinals nor a set of all cardinals. They cannot be established as subsets of V, since V is no set (and thus Separation does not apply to it).

For V to be more than a stopping point to be superseded V has to be an entity *sui generis*. This means informally that V is exactly what the picture of the iterative hierarchy shows it to be. V is *determined*, not indefinite, and *unique*. Formally this means

- that V cannot be an element of whatsoever other collection – on pains of re-introducing distinctions of the set/class-type
- that there are no other entities of V's type (not a collection of proper classes)
- that V is an entity which can be talked about by its name, without including it into a domain of reference.

V is not a standard object of (set theoretic) model theory. The only thing V 'does' is containing all the sets. A universally quantified sentence of pure set theory is meaningful as there is an entity which provides all the variable values: V.

A unified language has to distinguish urelements, sets and V. Again: V cannot be unified with them in a domain. The name "V" refers to V rigidly. End of story.

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25 At some time, Cantor considered distinguishing several 'absolutely' large, 'inconsistent' collections (like those of all ordinals or all cardinals). But they play no role in a transfinite set theory based on standard logic. Even apart from producing antinomies, these collections play no indispensable role in proofs about sets. So, Cantor came to consider *the single* absolute, inconsistent totality beyond any further increase.

26 Cf. Moore 1990, pp. 114-22; Tiles 1989, pp. 95-107. The principle sometimes – ignoring Cantor? – is discussed as 'All-in-one Principle', going back to (Cartwright 1994).

Given the difficulties in understanding an incomplete universe and the fundamental role of a Domain Principle, why don't we just talk about V without assuming it to be the value of a bound variable? This appears reasonable as doing otherwise land us in an incomprehensible framework of indefinite existents.

Assume we do not give up on the infinite, whether we are Platonists or fictionalists or whatever else. There are then infinite collections. Comprising within them *all* of a kind not collectable item wise by finite beings. We collect them using our concept COLLECTING. If we talk about the *F*s we naturally assume that there is a collection *F* where they are in. Cantor's Domain Principle expresses this idea that the *F*s we quantify over or talk about can be collected into a totality. Sometimes the totality has to be of another type to avoid antinomies (e.g. in the set/class-distinction). As there seems to be no limit to this procedure we always progress to a wider domain.[27] The Domain Principle thus enforces the idea of the incomplete universe. A domain is added to the objects, giving a larger domain, which is added to the objects – and so on.

Unless, that is, we meet a fixed point in this progression. Informally, the totality of things to be thought of or to be talked of can be thought or talked of: it belongs to the very domain it defines. Thinking of 'the domain x is in' applied to it leaves us at it. Thus, it may be called a fixed point of the Domain Principle.

As intuitive as the Domain Principle may occur to us, leading us up the ladder of the indefinite may be too much, as we have seen above. We might accept that the whole construction has a limit: a collection *beyond* further collecting. There lays the naturalness of Limitations of Size: There is *one* size too big to be collected into a set. This collection better not be the set U to avoid severe complications in set theory, otherwise rather intuitive. So one may see the idea behind Limitations of Size *without* endorsing **NBG** or **MK,** or any other set/class-theory. Nothing is gained by having (several) classes. With a collection of classes the question of their collectability immediately arises.

The single limit object V might be different.

If that limit object V exists – neither a set, nor an extended set like an inaccessible cardinal, nor a class – **ZFC** is consistent. And if our intuitive notion of set rather endorses the General Continuum Hypothesis, we add it

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[27] Recently Rayo and Williamson (2003) and others have argued for 'unrestricted First-Order languages', i.e. for quantification *without* a domain. The formal proposal, however, must employ **SOL** and a richer meta-language for which similar problems arise. One may also consider the employed **SOL** as critical and problematic (cf. Weir 2006). Unrestricted quantification is only unrestricted beyond an object/meta-language distinction.

as well: **ZFGCH** is consistent, if V exists. Our notion of set suits V, and *vice versa*.

This conception of V as collecting all the sets but being a special limit object may correspond better to our concept SET than taking set theory just as the realm up to the first strongly inaccessible cardinal. [The Cantor quote setting the theme may illustrate this perspective.] Someone might argue that our concept SET takes us thus far, but that there are other mathematical objects and theories (especially those of large cardinals, measures etc.), which pick up the baton where **ZFC** hands it over. Although this sounds like a nice division of labour, the large cardinals are too set-like to provide a natural boundary to our concept SET, supposing it to fit to **ZFC** in the first place. V is a stop point, the first inaccessible cardinal is not. And large cardinals – again – give rise to the question where their hierarchy is collected in, inviting and requiring V or some V', landing us again in an incomplete universe. V is not an incomplete universe at all: although we cannot walk, count or 'powerset' us up to it, V contains all sets; they are not in the making, there are no processes of indefinite extension going on. In this respect V as an object at the limits of thoughts differs from the row of experiences discussed by Kant in the *Critique of Pure Reason*: Kant traces the antinomies to their common error of taking the series of experiences, which is only given piecemeal and prospectively ('aufgegeben') as a 'given' totality. As experiences are obviously under temporal construction their series can never be united – by whom? In an experience? Sets, in contrast, are not (temporally) constructed and thus should be collectible in a unity. Thus far we are carried by the Domain Principle. At that *limit* we 'simply' have V as an object, and stop adding it to a domain.

Our concept of SET may force stronger set theoretic axioms on us. This shows, however, not the incompleteness or growing extension of V, but the incompleteness of a theory like **ZFC**. Urelements and ∅ have no members, but are members; sets have and are members; V has members, but is not a member: it occupies a slot in conceptual space.[28]

We may say: our idea of V is an idea contained and connected to our concept SET. The special nature of V is forced upon us by the unfeasibility of the idea of an incomplete or thought independent but growing set theoretic universe. We know of V by the picture we have of the iterative hierarchy and the structural relations between the ranks.

This conception of V follows some intermediate path between the two ontological traditions in analytic philosophy. On the one hand there are reasons of conceptual analysis why V suits our concept SET. On the other hand, some peculiar postulates need to be laid down for V. "V" is a rigid

[28] The slot of neither having members nor being a member finds no existing filler, if there is not David Lewis' atomless, uncollectible 'gunk'.

designator naming an entity which does not belong to some domain of quantification, although all other entities and referents of names do!

The major difficulty here would be to allow for a level or form of meta-theory when talking *about* V which is outside of any *formal* system. That way may lay ineffability or some version of ontological semantic mystery!

Comparing noneism, the incomplete universe and the thesis of V being an entity *sui generis*, the third idea comes out best in its combination of conceptual analysis and axiomatic ontology. If an inconsistent or noneist ontology is too much to swallow when taking on such a paraconsistent system, then we have to opt for at least partial fictionalism with respect to (some) entities proposed within paraconsistent set theories. Then the exploration of universality in set theory naturally awaits a further thorough exploration of fictionalism. Too many difficult questions wait there: Fictions like fictional characters in literature depend historically and genetically on their authors, and maybe on still existent copies of the literary work and living readers (cf. Thomasson 1999), nothing of this sort can be said of pure sets. Pure sets (like in **ZFC**) are not just presented as abstract entities outside of space and time, but their presentation (the story told by **ZFC**) arguably does not depend on any particular set theorist – not even Cantor, Frege or Zermelo. There might be several intermediate ontological categories between such purely abstract entities and spatio-temporal entities (cf. Thomasson 1999, pp. 120-33). Even in **ZFCU** one may wonder about the singletons of contingent urelements like the Cologne Cathedral: It seems bizarre to assume it to exist before the building was finished or even planned, thus this *set* seems to have a historical place! The recent interest in fictionalism may lead to increased ontological options.

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