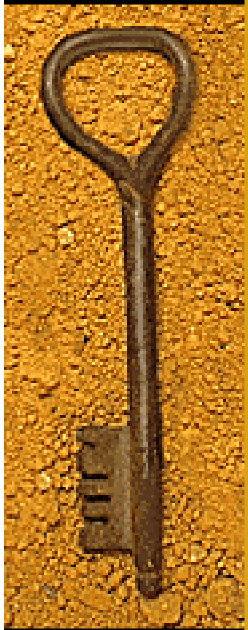


$p \wedge \neg p$

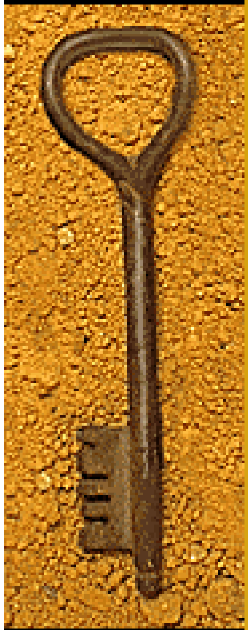
Manuel Bremer
Centre for Logic, Language and Information

Set Theory

- Apart from semantic closure set theory is one of the main motivations for the strong paraconsistent approach. As well as we take convention (T) to be basic for truth so do we take the naive comprehension scheme to be basic to our concept of set:

  (COMP) $(\forall F)(\exists y)(\forall x)(F(x) \equiv x \in y)$

- There should be a set corresponding to an open formula ascribing some property to the objects.
- This comprehension axiom is sometimes called "naive set theory". It was by no means present in all approaches that introduced set theory in the 19th century [Cantor‘s original set theory was different, concerned with combinatorial multitudes.; but it was present in the logicist approach of Frege and Russell]. "Naive Set Theory" lead to Russell‘s antinomy. So in today‘s standard set theory (**ZFC**, Zermelo-Fraenkel Set Theory with Axiom of Choice) there is no unrestricted comprehension axiom or schema [the (COMP) *schema* does not quantify over F].
- Paraconsistent set theory regains (COMP).



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Russell‘s Antinomy

- Russell‘s antinomy (usually called "Russell‘s Paradox") results from defining a set of all sets that are not elements of themselves:

  $R := \{x |\ x \notin x\}$

- The formula/property occuring in (COMP) can be any open formula. So it is not forbidden to take "( ) $\notin$ ( )". We get then:

  (1) $(\exists y)(\forall x)(x \notin x \equiv x \in y)$

  The y containing all the x that are not members of themselves is the set R, so we get:

  (2) $(\forall x)(x \notin x \equiv x \in R)$

  Now, what holds for any x (i.e. any set) has to hold for R as well:

  (3) $R \notin R \equiv R \in R$

  An antinomy!

- One can also start reasoning from the definition of R: If R is a member of itself, then R is (by R‘s definition) not a member of itself. If R is not a member of itself, it is (by R‘s definition) a member of itself. Contradiction!

Avoiding Russell's Antinomy in **ZFC**

- Russell's antinomy can be avoided by modifying (COMP) as **ZFC** does by restricting *separation* to an underlying set:

  (SEP) $(\forall F)(\forall y)(\exists z)(\forall x)(x \in y \wedge F(x) \equiv x \in z)$

  (SEP) or the corresponding *schema:* $(\exists z)(\forall x)(x \in y \wedge \varphi(x) \equiv x \in z)$ says that we can built a subset of any set given some condition *F*. (The right to left direction is crucial. By (SEP) **ZFC** can introduce $\varnothing$.)
- Now consider a kind of Russell set again. $R' = \{x | x \in a \wedge x \notin x\}$ for some arbitrary set a. We take substitution steps like before, and get:

  (4) $x \in a \wedge x \notin x \equiv x \in R'$
- Is R' an element of a? Either $R' \in R'$ or $R' \notin R'$. Assume $R' \in a$ and the first option, $R' \in R'$. Then we get (by (4) right to left direction):

  (5) $R' \in a \wedge R' \notin R' \equiv R' \in R'$

  Contradiction! Assume the second option, $R' \notin R'$. We get (by (4) left to right direction) another contradiction:

  (6) $R' \in a \wedge R' \notin R' \equiv R' \in R'$

  So both options lead to contradiction, and we have to give up the assumption $R' \in a$. The set a does not contain R', so a is not universal. Since a was arbitrary we conclude that *there is no universal set*!

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

No Universal Set in **ZFC**

- The argument given proves that there is no universal set in **ZFC**. This is mirrored in the way sets are built up in one of the models of **ZFC**, the so called "cumulative hierarchy" of pure sets (i.e. sets with no basic elements): one starts with the empty set $\varnothing$; the powerset of $\varnothing$, $P(\varnothing)$, containing all subsets of $\varnothing$ is $\{\varnothing\}$, since $\varnothing$ is a subset of every set (none of its elements in missing!); $P(\{\varnothing\})$ is $\{\varnothing,\{\varnothing\}\}$, since besides the empty set all singletons of elements of a set are members of the powerset; going on like that we introduce more and more sets, having k sets at stage n means 2^k sets at stage n+1 (at stage 6 we already have a number with twenty thousand digits!). We arrive at higher and higher cardinalities in a V shape figure. The universe of the cumulative hierarchy is therefore called V. V itself cannot occur in the hierarchy.
- Explained otherwise: There can be no universal set V, since P(V) would have more elements than V (by *Cantor's Theorem*), but that is impossible if V was the universal set. Contradiction!

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

What Is a Set?

- The standard account of concepts in **FOL** semantics goes like this: What does "( ) is a tree" refer to? It refers to the set of all trees. A concept/property is understood if we understand what its extension is.
- Now, what is a set? It cannot be the extension of "( ) is a set", since this extension would be a universal set in the cumulative hierarchy, but there is none. So in standard set theory there is no set/extension corresponding to our usage of "( ) is a set". For restricted usages (set of some kind, cardinality, order etc.) there are sets, but they do not speak of set in general. **ZFC** is *ontologically* restricted.
- The problem is the same as with the semantic hierarchy in the Tarski tradition [see Chap. 2]. Standard set theory is using a fundamental notion that cannot be explained *by this theory*! Or uses a fundamental notion that is *incoherent* given that very theory!
- Once again this runs against our intuitive understanding of "set", and seems to be a massive performative self-contradiction on side of standard set theory.

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

What Is a Consequence Relation?

- The absence of a universal set is not just a problem of understanding what a set is. Some set theoretical explanation of other concepts make use of a universal quantification about sets.
- If one defines:

  $\Sigma \models A$:= A follows from a set of premises Σ if and only if every interpretation that makes all $B \in \Sigma$ true makes A true.

  one now talks about *any* interpretation. And the domain of an interpretation is arbitrary. It may be a set of arbitrary high rank. So the supposed definition talks about *all* sets of an arbitrary high rank (i.e. of the completed hierarchy), but in **ZFC** we can never get at *all* sets! So it seems that our understanding of consequence cannot be modelled by **ZFC**. **ZFC** can only definite an incomplete model thereof.
- If there is no universal set, there is no universal complement of a set. Some theories (category theory) want to talk about such sets, however.

Using Classes?

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

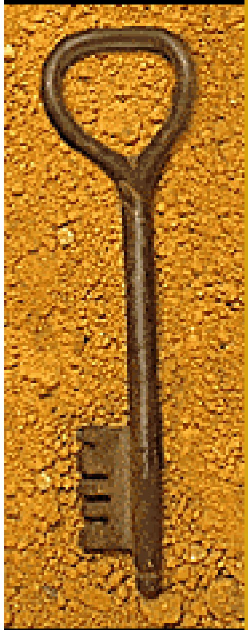
- In a version of standard set theory (von Neumann's version taken up by Bernays and Gödel) a distinction between classes and sets is used: Classes are sets that are not elements. The syntax distinguishes between classes and sets and gives a corresponding axiomatization.
- The supposed universal set turns out to be a class. The Russell set is a class, and so cannot be a member of itself.
- On the one hand this may be considered *ad hoc*, since the introduction of classes seems to play no further role than avoiding antinomies. And on the other hand there seems to be no reason why classes could not be elements of (some) other classes. The general inhibition of elementhood seems to be an exaggeration.
- If one then allows for classes to be elements of (some) classes we arrive at a second hierarchy (a hierarchy of classes), and now our old troubles return: There cannot be a universal class.

Extensionality

- In naive set theory the basic axiom besides that of Comprehension is that of Extensionality:

  $$\text{(EXT)} \quad (\forall x,y)(\mathrm{Set}(x) \wedge \mathrm{Set}(y) \wedge (\forall z)(z \in x \equiv z \in y) \supset x = y)$$

  Sets are identical if they have the same elements; if the domain does not contain any non-sets (i.e. in pure set theory) the first two conjuncts can be dropped. and the conditional be replaced by a biconditional.
- The predicate "Set( )" is problematic in **ZFC**, since its extension should be something like the universal set, absent in standard set theory.
- If (COMP) is used in its naive form it can be used to prove the existence of sets (like pair or power sets) that have to be introduced by axioms in standard set theory. Its basic supplementation is (EXT) alone!
- (EXT) can be used as an axiom, but often in paraconsistent set theory is given in form of extensionality *rules*.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Routley's Paraconsistent Set Theory

- Richard Routley uses the logic **DL** [see Chap. 5] with the following quantificational extension (Routley 1979a, p.290):
  Axiom schemes:

  (A14) $(\forall x)P(x) \rightarrow P(á)$

  (A15) $(\forall x)(A \rightarrow P(x)) \rightarrow (A \rightarrow (\forall x)P(x))$ *

  (A16) $(\forall x)(A \vee P(x)) \rightarrow (A \vee (\forall x)P(x))$ *

  (A17) $(\forall x)(P(x) \rightarrow A) \rightarrow ((\exists x)P(x) \rightarrow A)$ *

  [* x not free in A]

  Rules: (R3) $\vdash A \Rightarrow \vdash (\forall x)A$
- Comprehension is expressed with a Relevant conditional:

  (COMP') $(\exists y)(\forall x)(x \in y \leftrightarrow P(x))$
- (COMP') has no restrictions on "P( )" (like "y does not occur in P( )") so that one can have a set y such that $x \in y \leftrightarrow \neg x \in y$ (taking "$\neg(\ ) \in y$" as "P( )"). This is a bizarre set of all things that belong to it iff they do not belong to it!
- Such usages of (COMP') immediately raise the question of inconsistent ontology (i.e. *whether* and *where* are objects like the set described?).

Routley's Paraconsistent Set Theory (II)

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

- (COMP') is no longer well-founded: a set like $\{y \mid x \in y \leftrightarrow \neg x \in y\}$ may contain y itself; (COMP') allows for $\{x \mid x \in x\}$.
  [So the Foundation Axiom of ZFC is not part of paraconsistent set theory based on unrestricted comprehension. Foundation was a late comer in ZFC any way, has no mathematical applications outside set theory, and is dropped in otherwise standard non-founded set theory (cf. Aczel 1988).]
- Rephrasing (EXT) using a Relevant biconditional allows to derive even the Axiom of Choice from it, but also Irrelevant theorems, so (EXT) is replaced by a rule:

  (EXTR) (i) $x = y \Rightarrow x \in z \rightarrow y \in z$

  (ii) $x = y := (\forall z)(z \in x \leftrightarrow z \in y)$

  This rules allow for the substitution of identicals and defines how identity of sets is to be taken. For substitution we have:

  (SUB) $A \leftrightarrow B \Rightarrow C(A) \rightarrow C(B)$

  i.e. if A and B Relevantly imply each other than B can be substituted in any context C of A for A so that the resulting sentence is still Relevantly implied. (Different antinomies do not imply each other.)

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Routley's Paraconsistent Set Theory (III)

- (COMP') allows to define sets otherwise introduced by axioms:
- Existence of the empty set $\emptyset$:

  $$(\exists y)(\forall x)(x \in y \leftrightarrow \neg x=x)$$

  This set is empty, since even in **DL** we have: $(\forall x)x=x$
- Existence of the *absolute* complement of some set x:

  $$(\exists y)(\forall z)(z \in y \leftrightarrow \neg z \in x)$$

  x is some arbitrary set here, so that we can have the complement of any set we wish. In ZFC there are only relative complements of x (in some superset) because of the restricted version of (COMP).
- The antinomies can, of course, be derived in this set theory, but the underlying paraconsistent logic avoids triviality.
- That versions of the Axiom of Choice can be derived within his set theory shows, according to Routley, the *realistic* character of paraconsistent set theory (i.e. the domain is simply there, with all functions defined on it, whether we have constructed them from previously constructed material or not).

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Routley's Paraconsistent Set Theory (IV)

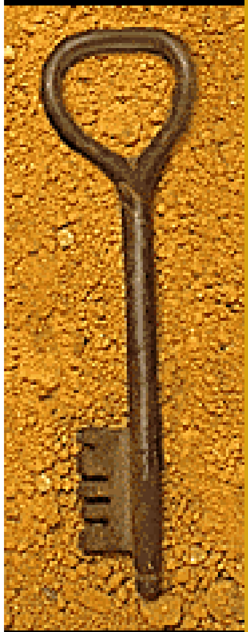
- What about the metalogic of Routley's set theory?
- Routley (1979a, pp.931-33) extends his logic **DKQ** by some axioms for arithmetic to his paraconsistent arithmetic **DKA**. He can prove that **DKA** is not trivial, i.e. absolute consistent.
- This proof, however, uses a truth functional conditional like that of **LP**. So this proof – because of the Curry Conditions – cannot be extended to paraconsistent set theory.
- A system like **DKA** being inconsistent does not meet the condition of *Gödel's Second Theorem*, so can be used itself to prove its own consistency.
- Routley and Brady (1989) nevertheless proved the non-triviality of a paraconsistent set theory using a logic with a negation semantics in terms of the Routley * and the ternary accessibility relation. Brady improved on that situation by stating his set theory in his logic **DJ**^{d}**Q.**

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Brady's Paraconsistent Class Theory

- Brady's version of paraconsistent set theory also employs (COMP') and (EXTR). The underlying logic is $\mathbf{DJ^{d}Q}$.
- Given the semantics of content containment Brady has to use (EXTR), since the *content* of x=y seems not to contain $x \in z \leftrightarrow y \in z$.
- In facts Brady's theory is a *class theory* (not a set theory). Brady distinguishes classes for which the two axioms hold and for which sentences dealing with them have a Relevant logic from sets, whose memberships sentences obey standard logic! The standard behaviour is needed to have enough countable sets in the classes.
- Brady has proved his system of class theory to be non-trivial.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Brady's Resolution of Antinomies

- Brady's version of paraconsistent set theory does not contain all of the antinomies and "only" keeps them from spreading triviality elsewhere. Some of the antinomies do not occur.
- In case of the Russell set one can prove $R \in R \leftrightarrow R \notin R$. To get to the explicit contradiction $R \in R \wedge R \notin R$ once needs either the Law of the Excluded Middle or Negation Introduction. Both are absent in $\mathbf{DJ^dQ}$. Thus given the validity of $R \in R \leftrightarrow R \notin R$ only one can chose to make them both true or both false.
- Something similar holds for Curry's Paradox, since Contraction does not hold in $\mathbf{DJ^dQ}$.
- Since Brady distinguishes sets from classes he restricts the validity of *Cantor's Theorem* (that the powerset of a set has more elements than the set itself) to sets, avoiding the antinomy that the powerset of the universal set has to be within the universal set and at the same time larger than the universal set. (The sets are collected into a class.)

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Adaptive Paraconsistent Set Theory

- One may try to gain further from the adoption of an adaptive logic by extending it with the two basic set theoretical rules. Ideally the resulting set theoretical logic should combine the basic power and many of the results of straightforward paraconsistent set theories (like Routley's or Brady's) with a severe restriction on reasoning with or multiplying inconsistent objects.
- The base logic may be the adaptive version of **LP** (Priest's so called "Minimal Inconsistent LP", introduced in chapter 7) with standard quantificational extensions giving **ALPQ** (Adaptive **LPQ**). Since set theory needs identity one in facts needs $\mathbf{ALPQ}^=$. $\mathbf{LPQ}^=$ however has a very weak concept of identity [cf. Chap. 4].
- The base logic considered here, therefore, is a set theoretical extension of the logic **UL4** as (to be) developed in chapter 20. **UL4** combines the extensional virtues of "Minimal Inconsistent LP" with a free quantification and description theory and identity rules which restrict substitution with inconsistent objects.

[The reader may return to these points here after having looked at chapter 20].

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Set Theory with **UL4** – Vocabulary

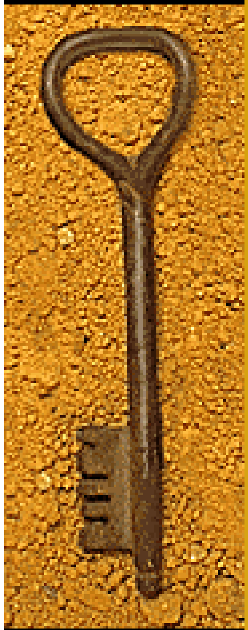
- We add to the constants of **UL4** the expression "Set( )" with its obvious intended meaning, as well as the usual set theoretical symbols like curly brackets, "$\in$", "$\subseteq$" etc. and "|" to express set abstracts like $\{x \mid P(x)\}$. We have to add the usual definitions like:

  (D$\subseteq$) $a \subseteq b := (\forall x)(x \in a \supset x \in b)$
  (D<>) $\langle a,b \rangle := \{a,\{a,b\}\}$

  Thus "$\in$" is taken as primitive.
- **UL4** deals not only with sets, so we need a set predicate "Set( )" to sort out the empty cases with non-sets, i.e. since in the intended interpretation $x \in y$ will be false if y is not a set, we have to avoid counterintuitive consequences of Irrelevant conditionals (e.g., $(\forall x,y)(x \in a \equiv y \in b)$ is true for any individuals *a* and *b* without these being identical).

  We use an introduction rule for some of these cases:

n.<o>	$á = \{x \mid P(x)\}$	…	Γ
m.<o>	Set(á)	(Set),n	Γ



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Set Theory with **UL4** – V and ∅

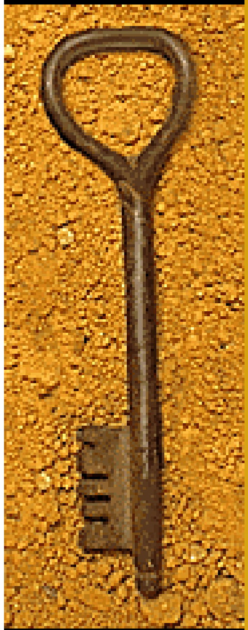
- We define the *universal set* V

  (DV) $V := \{x \mid x = x\}$

  Because of (=I) this is truly the universal set.

- Since we cannot exclude inconsistent objects like a with $a \neq a$ it is no option to define ∅ as $\{x \mid x \neq x\}$!
  A better idea is:

  (D∅) $\varnothing := \{x \mid x \notin V\}$



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Set Theory with **UL4** – Extensionality

- Extensionality of sets can be added as a rule to introduce identity of sets.

n.<k>	$Set(a) \wedge Set(b) \wedge (\forall x,y)(x \in a \equiv y \in b)$	…	Γ
m.<k>	$a = b$	(Ext),n	Γ

- (=E) doing the rest for consistent sets, since both sides of "$\in$" are open for substitution. Reasoning with inconsistent sets may be impaired by (=E), but it is far from clear whether this is a disadvantage [cf. Chap. 17].

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Set Theory with **UL4** – Comprehension

- Naïve Comprehension is added as rule/axiom schema using the material conditional:

  $$\text{n.}\langle\ \rangle \qquad (\exists y)(Set(y) \wedge (\forall x)(x \in y \equiv P(x))) \qquad (NC) \qquad \emptyset$$

  with no further restrictions.
- (NC) immediately gives us the existence of $\emptyset$ and V.
- Given the restrictions on detachment in **UL4**, however, may forbid or retract the application of detachment here. Consider, for example, the inconsistent object *a* with $a \neq a$. We have

1.< >	$a \in V \equiv a = a$	(NC), (DV), ($\forall$E)	$\emptyset$
2.< >	$a \in V$	($\equiv$E), (=I), 1	$\{°a=a\}$
3.<3>	$a \neq a$	AE	$\emptyset$
4.<3>	$a \neq a \wedge a = a$	($\wedge$E), (=I), 3	$\emptyset$

  Now, given (4) "a = a" obviously isn't consistent, so (2) has to be retracted! That, of course, does not mean that $a \notin V$.

p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

Set Theory with **UL4** – Virtuality

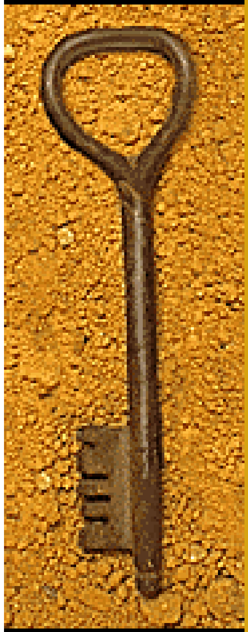
- **UL4** contains non-referring singular terms, using a name does not imply that the named object does exist.
- Singular terms naming sets – including set abstracts! – thus need not refer by just being singular terms. In principle there would therefore be the option for *non-existent sets* being around. These may one remind of Quine's "virtual sets" (Quine 1963). Virtual sets, with Quine, are set abstracts which are not quantified over, thus not being said to exist. **UL4** could provide a place for such virtualities.
- On the other hand (NC) just declares that any set whatsoever (i.e. any set defined by a set abstract) exists.
- One may consider whether the introduction of a second set of quantifiers (quantifying over *possibilia* or *virtualities* as well) might be useful, using a quantifier with no existential impact in (NC). Apart from the problems of an ontology of *virtualities* this seems, to me, to be against the spirit of Naïve Comprehension, the very point of which seems to be that there (really) *is* a set to each defining condition.

Dunn's Negative Result

- Michael Dunn showed a somewhat disturbing result for non-classical logic and thus for non-classical set theory:

  > If one combines a couple of basic and innocent principles for the classical connectives and for the consequence relation (like transitivity) with second order quantification and conversion principles (for a λ-calculus like abstraction) *then* the resulting logic (i.e. the set of theorems generated) is an *extension* of *classical* Second Order Logic.

- This means that the full strength – and paradox yielding power – of classical reasoning (including Disjunctive Syllogism …) is regained.
- The systems by Routley and Brady outlined in the preceding slides and versions of set theory framed in first order **LP** or **UL4** are – as first order systems – not fulfilling the antecedents of Dunn's theorems.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

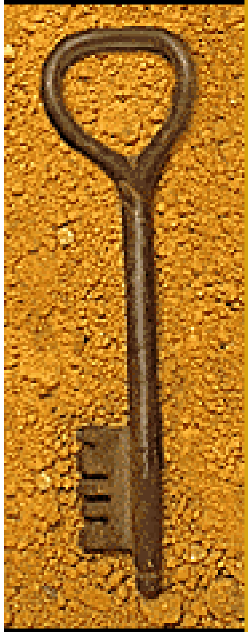
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Systematization of Antinomies

- Within paraconsistent set theory the general form of many antinomies can be given [cf. (Priest 1995, pp.142-48); based on (Russell 1905)]:
  - (i) $\Omega = \{y \mid \varphi(y)\}$ exists and $\psi(\Omega)$
  - (ii) if $x \subseteq \Omega$ such that $\psi(x)$:
    - (iia) $\delta(x) \notin x$
    - (iib) $\delta(x) \in \Omega$
- If we have some set Ω of the y satisfying some condition φ and that set satisfies some condition ψ, we get an antinomy if there is some subset of Ω having also the property ψ, and there is some diagonalisation function such that the diagonal of x is not in x itself, but has to be in Ω.
- The antinomy occurs by taking Ω itself for x, getting:

$$\delta(\Omega) \in \Omega \wedge \delta(\Omega) \notin \Omega$$

- In case of Russell's antinomy Ω is the universal set, φ is the condition of being a member of the universal set. ψ is not important in that antinomy and can be selfidentity. The diagonalisation δ concerns not-membership. $\delta(\Omega)$ is the Russell set $\{y \in \Omega \mid y \notin y\}$. $\delta(\Omega) \in \Omega$, since it is a set, and $\delta(\Omega) \notin \Omega$, otherwise $\delta(\Omega) \in \delta(\Omega)$. Contradiction in any case.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

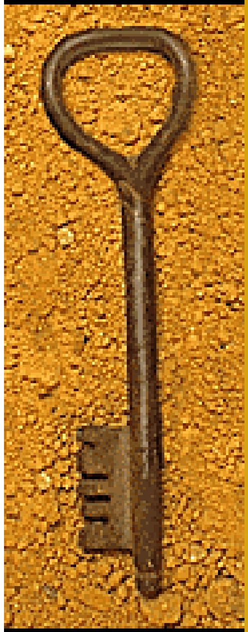
Systematization of Antinomies (II)

- Taking this "inclosure schema" yields the following overview:

Antinomy's Name	Set Ω	Condition φ	Property ψ	Diagonal $\delta(x)$	Antinomy
Russell's Antinomy	Universal Set V	y is a set	$z = z$	$\neg x \in x$	$R \in R$ and $R \notin R$!
Burali-Forti Antinomy	Set of the Ordinals	y is an ordinal	$z = z$	next greater ordinal	greatest ordinal is no ordinal!
Miroma-noff's Antinomy	cumulative hierarchy V (of well-founded sets)	y is well-founded	$z = z$	being contained in V	$R \in R$, since all subsets of R are, and $R \notin R$!
König's Antinomy	definable ordinals	definable	being indefinable	smallest ordinal $y \notin \Omega$	$y \in \Omega$, *since* defined, and $y \notin \Omega$

Miscellaneous Remarks

- Paraconsistent set theory is not only of interest in itself or as formalization of naive set theory. It also may serve as the foundation of paraconsistent arithmetic.
- It has to be checked what remains of standard mathematics once its foundations in set theory have been restricted to paraconsistent set theories!



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Assessment

- Do we need paraconsistency for set theory?
- If the justification for a paraconsistent set theory depends on avoiding the antinomies, others approaches that avoid the antinomies might be alternatives.
- If the justification depends on the argument that the notion of set is not clear unless we have a universal set, set theories that combine standard logic with having a universal set might be alternatives. [e.g. Quine's New Foundations (**NF**) set theory, (Quine 1937), or (Church 1974)]
- A more decisive point may be keeping unrestricted (naive) Comprehension.
  That is something that neither **ZFC** nor systems like **NF** can do.
- Neither does the distinction between classes and sets capture our naive understanding of sets. A theory not making this difference should be preferred.

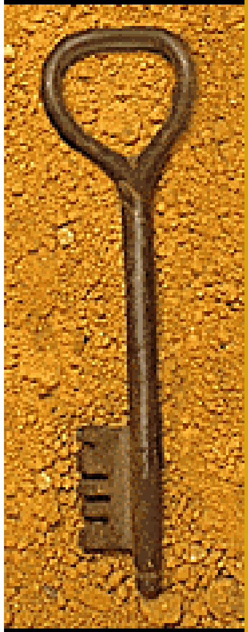
[cf. Chap 18 on all these points]

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information
HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Questions

- (Q1) Does a comprehension schema that allows for non-wellfoundedness meet our naive intuitions about sets? Do we usually take sets to be well-founded?
- (Q2) One of the major problems of Russell's and Frege's logicist programme (of reducing mathematics to logic) was that some axioms of set theory look very substantial and not purely logical. The main concern of Russell is the Axiom of Infinity (*postulating* a set with infinitely many elements). What is the upshot of paraconsistent set theory for Logicism?
- (Q3) Why does $\emptyset := \{x \mid x \notin V\}$ work?
- (Q4) Why has "Set(y)" been added in **UL4** Comprehension?



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Exercises

- (Ex1) Show by defining them that Pairing and Powerset are not needed as axioms in Routley's set theory, but are instances of (COMP').
- (Ex2) Take Berry's Antinomy of the smallest number z that cannot be named by less than 99 words. Show how this antinomy fits the inclosure schema.
- (Ex3) Consider the inconsistent object a with $a \neq a$. Why can one not (finally) derive $a \in \varnothing$ having proven $a \notin V$?

$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Further Reading

- On the problem of the set hierarchy see Priest‘s *In Contradiction*, pp.35-47.
- On **ZFC** see a standard text book like: Stoll, Robert. *Set Theory and Logic*. New York, 2nd Ed., 1963. On the different versions of standard set theory and models for **ZFC** see: Lavine, S. *Understanding the Infinite*. Cambridge/MA, 2nd Ed., 1988.
- On paraconsistent set theory see besides the usual references (cf. Chap.1) see the work of Richard Routley (1979a, pp. 914-25) and Ross Brady (2000).
- On Dunn’s theorems see his “The Impossibility of Certain Higher-Order Non-Classical Logics with Extensionality”, in: D. Austin (Ed.) *Philosophical Analysis*, Dordrecht, 1988, pp. 261-79.

- Compare with Chap. 2 on the problem of semantic closure. See the discussion of Lesniewski‘s solution of Russell‘s antinomy and other non-standard set theories in Chap. 18.