$p \wedge \neg p$

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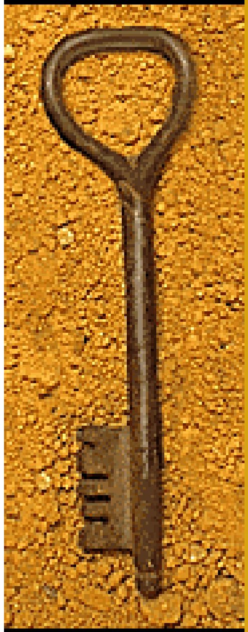
Relevant Logics

- Relevant Logics (RL, in America also called "Relevance Logics") are one of the major groups of paraconsistent logics. Relevant logics are a subset of the paraconsistent logics (i.e. the requirements to be Relevant are stronger than only avoiding Explosion).
- The essential aim of Relevant Logic is to capture a natural and intuitive appropriate concept of entailment, one for which the paradoxes of implication (or similar ones) do not appear.
- The very earliest axiomatization of a paraconsistent logic by Orlov was in fact a Relevant logic. The development known today as Relevance Logic started in the USA in the 1960s with work done by Anderson, Belnap, Dunn and Meyer, and resulted in two volumes entitled *Entailment*. In the 1970s Routley, Meyer, Brady and others developed in Australia their version of Relevant Logic and semantics. This issued in two volumes *Relevant Logics and their Rivals*.
- We are interested in Relevant Logic here as a means to define a conditional that keeps *Modus Ponens* and its other essential properties, but does not fail on the conditions we proposed for paraconsistent logics (e.g. the Curry Conditions).



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Propositional Relevant Logics

- The defining properties of Relevant Logics concern the propositional level, so we will focus on propositional systems here [cf. Chap. 4].
- [At the end of this chapter we introduce the quantificational logic **DJ^{d}Q**, since we will need it as a quantificational logic in the chapter on set theory.]

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Defining Relevant Logics

- RLs not just avoid Explosion, all Irrelevant theorems and consequence relationships of **PC** (and standard modal logics with an Irrelevant strict implication) should be avoided.
- These include all versions of Explosion like
  (1) $A \supset (\neg A \supset B)$
  (2) $A \wedge \neg A \supset B$
  and all versions of *verum ex qoudlibet sequitur* like
  (3) $A \supset (B \supset B)$ or
  (4) $B \supset (A \supset B)$ [resp. $B \vdash (A \supset B)$]
- What have all these sentences in common?
  (If there is one such property, excluding it will yield only Relevant implications.)
- The diagnosis of proponents of RLs is that in all these sentences the consequent ultimately to be detached is a sentence that has no connection in content to the antecedent sentences. That there is no connection in content can be seen in the propositional case by the fact that given the full logical structure at the propositional level the consequent and the antecedent share no propositional letters.

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Defining Relevant Logics (II)

- This observation led to the *variable sharing requirement* (speaking of a letter for a basic sentence as a "propositional variable").
- This leads to the following definitions:
  (R1) Only those conditional theorems are Relevant in which the antecedent and the consequent share a propositional variable.
  (R2) Only those consequence relationships are Relevant in which the premises and the conclusion share a propositional variable.
  (R3) Just those logics are Relevant which have only Relevant theorems and and Relevant consequence relationships.
- Requirement (R1) is necessary, since given the validity of *Modus Ponens* any Irrelevant theorem would spread its Irrelevance.
- Seen in the light of these definitions **LP** is not a Relevant logic if we consider "⊃" as its conditional.
- We will not discuss the adequacy and philosophical background of these definitions in detail here. We will see what sticking to these properties requires, and whether we are willing to go this way.

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First Degree Entailment

- Relevant Logics aim at a concept of entailment that is stronger than the material conditional.
- The first approach could be to keep the material conditional as the usual abbreviation of $\neg A \vee B$ and consider $\vdash$ as formalising entailment. $\vdash$, of course, is a stronger relation than "$\supset$". Alternatively once could require entailments $A \rightarrow B$ to be of first degree only (i.e. there are no nested arrows $\rightarrow$).
- This approach was taken in the late 1950s by Anderson and Belnap. The system is called First Degree Entailment (**FDE**).
- One can give a tree-characterisation of this system by rules similar to the rules we used in chapter 4 for **LP**; this system also allowing for truth value gaps. The resulting logic is in fact a sub-logic of **LP** (since **LP** allows for some Irrelevant deductions). **FDE** has no *theorems* at all! Its logic resides completely in the consequences that hold in it.
- The fundamental shortcoming of this system – besides being even weaker than **LP** – is that it does not allow to model nested entailments, whereas many intuitive logical truths for the conditional (Permutation, Transitivity etc.) are just that.
- [For a brief tree-based introduction to **FDE** see (Priest 2000, Chap.8).]

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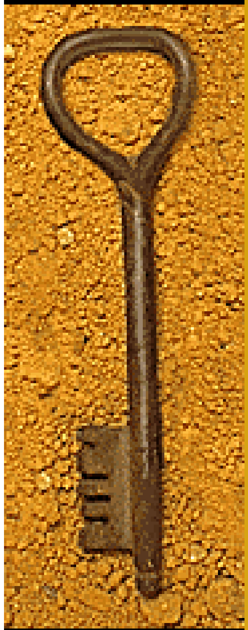
Example Relevant Logic

- Let us look at system with a nestable conditional connective "→".
- The system **DL** ("dialectical logic") is used by (Routley/Meyer 1976) in a paraconsistent context:
- Axiom schemata:

(A1) $A \rightarrow A$
(A2) $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$
(A3) $A \wedge B \rightarrow A$
(A4) $A \wedge B \rightarrow B$
(A5) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(A6) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
(A7) $\neg\neg A \rightarrow A$
(A8) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
(A9) $A \rightarrow A \vee B$
(A10) $B \rightarrow A \vee B$
(A11) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$
(A12) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$
(A13) $\neg (A \wedge B) \rightarrow \neg A \vee \neg B$

Rules:
(R1) $A, A \rightarrow B \Rightarrow B$
(R2) $A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$



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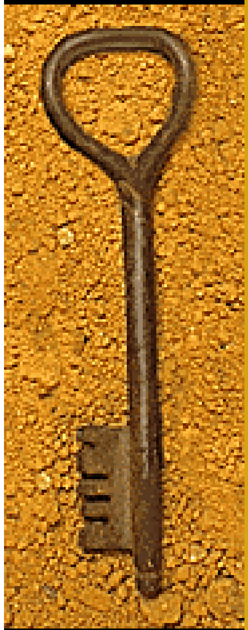
Example Relevant Logic (II)

- In these logics the consequence relation only inherits truth from the premises to the conclusion (and so it's done in a derivation). The conditional "→" expresses a stronger relation. It expresses the obtaining of some connection between the antecedent and the consequent. Because of this it is not valid that

  (*) $\vdash A \rightarrow B$ if $A \vdash B$

  The *Deduction Theorem* generally does not hold for entailment. [To have a *Deduction Theorem* the reading of $A_1,...A_n \vdash B$ has to be changed (taking the left hand side as a "multiset") and further connectives are introduced like an intensional conjunction (called "fusion") and disjunction (called "fision"), but all this is beyond our current interests.]
- As you see, these logics need quite more axioms than standard logics, since a lot of proofs are no longer available. For example distribution needs a special axiom (A6). We need two axioms, (A12) and (A13), to establish the DeMorgan dualities. With respect to negation (A7) shows its standard character. Note also that contraposition has to be taken as a rule of its own right.

No Binary Accessibility



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- The greatest challenge for Relevant logicians is to give a semantics that makes exactly the consequence of the axioms given true.
- Usually modal logics (i.e. logics that model a stronger conditional) use a possible worlds semantics. A typical truth conditional for a strict conditional (with the set of possible worlds W) is:

  $$(*S\rightarrow)\ v(A \rightarrow B,w) = 1 \text{ iff } (\forall w' \in W)(Rww' \Rightarrow (v(A,w') \Rightarrow v(B,w'))$$

  This rule makes A → A a logical truth. This is O.K., since (A1) says so, but it also makes B → (A → A) a logical truth, and this sentence is Irrelevant, since B and A → A share no propositional variables.
- The problem is combining the supposedly obvious truth of sentences like A → A with avoiding Irrelevance. A binary accessible relation seems to be unable to keep them apart.
- Therefore many Relevant logics have a *ternary* accessibility relation!
- To model inconsistency not only consistent, but also inconsistent worlds are considered, even worlds in which the laws of logic are thought to be out of force. (This is modelled by not applying the usual rules of the connectives to these worlds.)

Ternary Accessibility

- A model is a tuple *M*=<@, *N*, *W*, R, *, *D*, *v*> consisting of the actual world @, the set of *normal worlds* (those worlds in which logic holds), the set of all worlds (normal or not), the accessibility relation, the monadic *-operation, a domain of objects (in the quantificational case [we are not considering here]), and an interpretation *v*.
- We require:

  1. $@ \in N \subseteq W$
  2. $D \neq \varnothing$ (in the quantificational case)
  3. A is true in some Model M' iff $v'(A,@)=1$

This last requirement states that we assess the truth of a sentence relative to the actual world only. All other worlds are only operative in determining the truth conditions of a sentence. This may be justified by considering that we want to know which sentences are true, and a sentence is true (in distinction to its being logically true) if it is true in the obtaining world (i.e. the actual world).

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Ternary Accessibility (II)

- The semantic rules are the usual ones for the connectives with the exception of the intensional negation rule (see below) and the rule for the truth of entailments in terms of ternary accessibility:

$$(R\rightarrow)\ v(A\rightarrow B,w)=1 \Leftrightarrow (\forall w',w'')(R(w,w',w'') \wedge v(A,w')=1 \Rightarrow v(B,w'')=1)$$

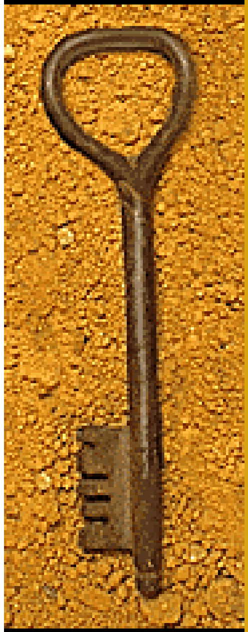
An entailment is true at some world if this world sees an accessibility between two other worlds such that if A is true at the first of these worlds B is true at the other.

- For normal worlds a further requirement is stated:

$$(NC)\qquad (\forall w,w',w'')(w\in N \wedge R(w,w',w'') \Rightarrow w' = w'')$$

This serves to recover the idea that under normal conditions an entailment just is true if in all accessible worlds the truth of the antecedent guarantees the truth of the consequent.

- The *normality condition* is important in its application to @ and in regaining a model of intuitive entailment as *normal implication*.



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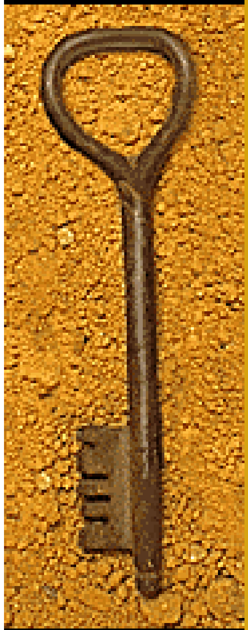
Ternary Accessibility – Example



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- As an example let us see why A→(B→B) is no longer a logical truth: Since we are free to interpret atomic formula at worlds and the truth condition (R→) refers to 3 worlds of evaluation, we can have *v*(B,w')=1 and *v*(B,w'')=0, being accessible from w''', thus B→B being false at w''', thus with @ being the actual world of some model accessing a world w'''' where A is true and w'''' accessing w''', R(@,w'''',w'''), we have according to (R→) made A→(B→B) false!
- That B→B is false at w''' does not matter, since only @ matters for assessing truth in a model. B→B cannot be made false at @, since the normality condition (NC) has it that for a normal world the seen accessing and accessed world are identical.

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Some Binary Accessibility

- Further requirements re-introduce a *binary* accessibility relation $\leq$ for two worlds w and w':

  4. $w \leq w'$ iff $(\exists w'' \in N)(R(w'',w,w'))$

  That is worlds are binary accessible only if some *normal* world "sees" the accessing.
- Given this definition in 4.

  5. $w \leq w$
  6. $w \leq w' \wedge R(w',w'',w''') \Rightarrow R(w,w'',w''')$

  This preserves the seeing of accessibility from an accessed world to the accessing world.
- The next requirement ensures double negation:

  7. $w = w^{**}$
  8. $w \leq w' \Rightarrow w'^{*} \leq w^{*}$

  This last supports contraposition.

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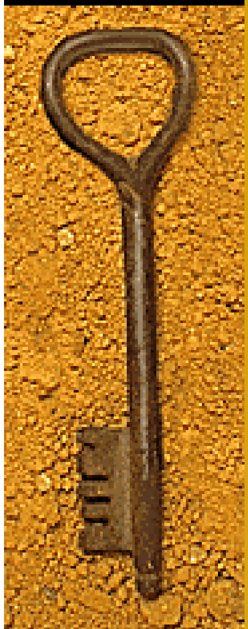
Some Binary Accessibility (II)

- " ≤" defines a *binary* accessibility relation in terms of R and normal worlds. One can uses *this* relation to model a concept of entailment that expresses that the truth of the antecedent necessitates the truth of the consequent. Thus we define a common notion of entailment:
  (DE) A *entails* B $\Leftrightarrow (\forall w \in W)(v(A,w)=1 \Rightarrow v(B,w)=1)$
  To achieve expressing this within Relevant semantics using the ternary R and the binary ≤ we add a further heredity requirement:
  9. $v(A,w)=1 \wedge w \leq w' \Rightarrow v(A,w')=1$
  and we define *normal implication* as:
  (DNI) A *normally implies* B $\Leftrightarrow (\forall w \in N)(v(A \rightarrow B,w)=1)$
  Normal implication thus is Relevant implication in the normal worlds only. Interestingly enough now one can prove an important meta-logical theorem:
  (MT1) A *entails* B iff A *normally implies* B.
- The non-normal worlds, however, are *essential* to Relevant semantics (to invalidate Irrelevant theorems for "→"). We forbid that the actual world of a model is a non-normal world, but maybe we shouldn't!
- So the set of normal implications need not match the set of *true* conditionals A→B of a model (and, of course, of the logic as a whole).



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Ternary Accessibility?

- (MT1) expresses that given normality the Relevant entailments coincide with our "usual" entailments (of binary accessibility modal logics). Relevant entailment itself, however, is different.
- What does (R→) tell us?
  This seems quite unclear. No intuitive reading is obvious. But if the semantics is not justifiable with respect to our intuitive concept of entailment, the whole approach using a ternary accessibility relation seems questionable. Even Relevant Logicians [cf. Anderson/Belnap/Dunn 1992, pp. 163-64] summarize that (R→) *has no intuitive basis* and no connection to our ordinary understanding of the conditional!
- Even the definition of the binary accessibility relation ≤ (cf. requirement 4) has no intuitively obvious justification.
- There is, however, a new promising reading of the ternary accessibility relation in terms of the theory of information (flow).

[Before we look at this we look at the Routley semantics of negation.]

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The Routley Star

- To give a semantics for negation that allows for true contradictions Richard Routley introduced his (in)famous star operator. Negation becomes an intensional connective.
- Each world w is accompanied by a *witness* world w*.
- The rule for negation is:

$$(*\neg) \qquad v(\neg A, w)=1 \text{ iff } v(A, w^*)=0$$

- Once again the evaluation of a sentence is spread across worlds. Note that nothing is said whether the behaviour of A (i.e. unnegated) depends on what is the case at w*.
- Example:
  We can have $v(A,w)=1$, $v(A,w^*)=v(B,w)=0$ giving us that
  $A \wedge \neg A \rightarrow B$ turns out false, since w is a world where A and ¬A are true (i.e. being non-normal) and B isn't. Explosion fails.

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The Routley Star (II)

- It is hard to see any intuitive sense in the witness world and the rule for negation.
- The construction validates the theorems, but it has been considered by many logicians as a formal trick only [for example, cf. (van Bentham 1979)].
- As in the case of the ternary accessibility relation there is also a new reading of the Routley star in terms of information theory based on situation semantics.

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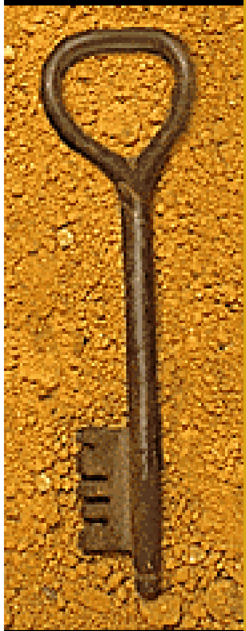
Information Flow Semantics

- A more justifiable semantics for negation and the relevant conditional can be given in the framework of information flow (the theory of distributed systems and information channels) based on the framework of situation semantics.

- [It is not possible to throw in quickly an introduction to that field as well. There are some other Powerpoint based lectures, however, that do that. See http://manuel.bremer.bei.t-online.de/situationsemantics.htm on situation semantics in general. And see on information flow and channels http://manuel.bremer.bei.t-online.de/BarwiseSeligmanESSLII.htm]

- Information flows in *distributed systems* (like a circuit that connects a switch with a bulb). These systems can be considered *channels* along which we reason (like knowing that the switch was pressed if the bulb is burning).
- Talking in situation semantics language we can say that information about one *situation* is derived from another situation by some channel.
- Situations are parts of reality. The channel is also a part of reality (trivially so if we see the distributed system as a channel). Therefore there is a situation that comprises exactly the channel.



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Information Flow Semantics for →

- The infons made factual by that situation are the infons that describe the structure of the channel. The important thing is: The channel as well as the parts (be it two or more) are situations and can be relata of a ternary relation connecting entities of some type. The channel and two sites *connected by a channel* can be seen *as the relata of the ternary accessibility relation* R. Look again at the truth condition of "→": The relation that is said to hold between w' and w'' is established or observed from w. Let w be the channel and w' and w'' be the sites connected by the channel. We can say that the channel establishes the connection between the sites, and from the perspective of the distributed system the sites are relata of information flow.
- A→B receives the following interpretation: A→B is true at w if all the sites/situations w' and w'' connected by that channel are such that if the information A is available at the one end w the information B is available in situation w''. The more constraints hold for a distributed system *the less* situations can be derived from it, since the derived situations have to meet all the constraints. Having more constraints operative in a channel means zooming in on a very specific part/situation.

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Information Flow Semantics for → (II)

- This amounts to an interpretation of a ternary accessibility relation in terms of information flow. And it allows us to introduce a second reading of "→" (besides the usual one).

- (R→2) A→B is true with respect to a channel *c* iff
  $\{A\} \vdash_c \{B\}$ is a constraint for that channel *c*.

- To spell out this reading there should be infomorphisms mapping what is said in A and B at the system level to statements/infons of the part classifications. An interpretation also has to be developed for the equivalent of the occurrence of nested arrows "→". It seems that one has to allow constraints that constrain lower level constraints.

- Edwin Mares gives a further interpretation of "→" in terms of situations and information flow exploiting the fact that an infon (made true by some situation) can contain information about situations (i.e. contain parameters or constants for situations).

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Information Flow Semantics for → (III)

- An infon can contain information about relations between infons and situations. Infons of this kind are *informational links*. They tells us that some infon being factual *involves* a second infon being factual. An informational link has the form: <<*involves*, Φ, Ψ, 1>> where Φ and Ψ are situation types.
- One can take R to represent these informational links. If some informational link is said to hold in situation w saying that an infon ϕ (of type Ψ) carries the information that the infon φ (of type Φ) also has to hold (somewhere), then if R(w,w',w'') and w' contains/supports the infon ϕ, then w'' contains/supports the infon φ. The ternary relation models an informational link of one situation/infon constraining another. Thus (taking → here for a moment as connecting infons):

(R→3) $\phi \rightarrow \varphi$ is true with respect to situations w, w', w'' such that R(w,w',w'') if it is true that if w$\models$ <<*involves*, Φ, Ψ, 1>> and w'$\models \phi$, then w''$\models \varphi$.

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Information Flow Semantics for → (IV)

- If we take R to be about or related to informational links some features of information linkage should be mirrored by the relation R. Every situation is closed under informational linkage, since information is factual. This means for relation R that one should require it to be reflexive: R(w,w,w)
- The set of statements made true by w is closed under *Modus Ponens* then.
- Note that reflexivity does not make obvious sense, however, in the second reading of "→": Not every channel carries information from itself to itself! On the other hand it seems vacuously true, since the information present in the system is present in the system (as its own improper part) if it is present in the system (as its own part).

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Information Semantics for *

- What about negation? If the situation framework could deal with the *-operator as well it might well become the preferred reading of Relevant truth conditions!
- A piece of information can be incompatible with another piece of information. This may be due to the polarities within the infons supported by the situation or due to constraints yielding implied information of the one infon incompatible with the other. Suppose we had made this intuition more precise and given a formal account of compatibility, say by using infon logic or by considering metalogical properties of the theories of distributed systems (i.e. their set of supported constraints).
- We could introduce then a relation C(w,w') of *compatibility* between situations or the sets of infons supported by these situations. The statement ¬A is true at w, then, if the truth of A is incompatible with the other information contained in a related situation w'. The closure of a situation can be taken as what the situation says. The set of compatible situations can be taken as that which is not excluded by what the situation says.

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Information Semantics for * (II)

- If the negation of some state of affairs A is a fact in all w compatible situation, the state of affairs is not compatible with w, so presumably its negation is contained within what w says. w* (the witness world) can be taken as the *maximal* situation such that the information given in w is compatible with it. We arrive at:
- (R*2) $v(\neg A,w)=1 \Leftrightarrow (\forall w')(C(w,w') \Rightarrow v(A,w')=0)$

  That is a negation is true in a situation w if in all other situations compatible with the information given in w the statement under consideration is false. w* just is the most informative (since most comprehensive) of these situations. *Not* every situation is compatible with itself, since we assumed that there are inconsistent situations.
- So a situation framework may also give us a more intuitive understanding of the *.
- Note, however, that this requires spelling out the relation C in an acceptable way. Sometims this relation is just taken as basic [e.g. (Mares 2004, p. 77)]!

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Worlds and Situations

- The information (flow) semantics outlined was developed in terms of situations.
- Situations – in contrast to possible worlds – are *partial*. They do not contain A or ¬A for every sentence A of the language. They are not – in case we don't require this otherwise – closed under some rules.
- Especially situations (in contrast to situation *types*) can be taken as parts of the universe. Taken thus it makes sense to say that some situations are connected or parts of a greater distributed systems.
- The same cannot be said for worlds. Worlds are not only total, only one of them is actual. Different possible worlds stand in no causal or part-whole relation whatsoever.
- Therefore it might be crucial to rephrase the whole semantics into talk about situations and situation types.
- This change to partiality might, however, conflict with the information semantics for *, since assessments of compatibility may require closure of the situations under consideration.
- So more work has to be done here. We have seen so far the idea of such a semantics. [More complications arise when the partiality of the situation framework is combined with quantificational semantics in an intensional framework of world relative domains.]

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Systematization

- The typical systems of Relevant Logic can be systematized as being extension of a basal Relevant logic **B**.
- B is defined by the following axiom schemes and rules:

(A1)	$A \rightarrow A$
(A2)	$A \rightarrow A \vee B$
(A3)	$B \rightarrow A \vee B$
(A4)	$A \wedge B \rightarrow A$
(A5)	$A \wedge B \rightarrow B$
(A6)	$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
(A7)	$(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(A8)	$(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$
(A9)	$\neg\neg A \rightarrow A$

(R1)	$A, A \rightarrow B \Rightarrow B$	[*Modus Ponens*]
(R2)	$A, B \Rightarrow A \wedge B$	[$\wedge$I]
(R3)	$A \rightarrow B \Rightarrow (C \rightarrow A) \rightarrow (C \rightarrow B)$	[Prefixing]
(R4)	$A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$	[Suffixing]
(R5)	$A \rightarrow \neg B \Rightarrow B \rightarrow \neg A$	[Contraposition]

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Systematization (II)

- The systems that extend **B** are called *affixing* systems, since they all share the pre- and suffixing rules.
- Candidates for further axiom schemes are:
  - (A10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
  - (A11) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
  - (A12) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
  - (A13) $(A \rightarrow ((A \rightarrow B) \rightarrow B)$
  - (A14) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
- (A10) – (A12) are the theorem versions of the rules (R3)-(R5). (A13) is a version of the *Modus Ponens* theorem, (A14) Absorption (or "Contraction").
- All of these are not valid in **B**. One can define the systems:
  - **DW = B** + (A10) [(R5) can be dropped then]
  - **TW** = **DW** + (A11) + (A12) [drop (R3) and (R4)]
  - **RW** = **TW** + (A13) [drop (A11) *or* (A12)]
  - **R** = **RW** + (A14)
- **R** being the strongest of these systems contains all PC-theorems! The interesting question is which theorems hold for "$\rightarrow$" (not "$\supset$").

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Restrictions on R

- The systems that extend **B** validate more sentences by excluding more countermodels by requiring further properties of the relation R (as stronger modal logics in general work by restricting accessibility).
- For example:
  - making Prefixing as an axiom true corresponds to:
    $(\forall w',w'',w''',w'''')((\exists w \in W)(R(w',w'',w) \wedge R(w,w''',w'''')) \Rightarrow (\exists w''''' \in W)(R(w',w''',w''''') \wedge R(w'',w''''',w'''')))$.
  - making Contraction as an axiom true corresponds to:
    $(\forall w',w'',w''')(R(w',w'',w''') \Rightarrow (\exists w \in W)(R(w',w'',w) \wedge R(w,w'',w''')))$
- Both this requirement on R have no intuitive sense and can hardly be rephrased to make them obvious. The affixing theorems, however, look pretty natural. Why do they demand such strange restrictions?
- These conditions on R also have most unfortunate consequences. They employ nested existential quantification. If one uses them in a tree like decision procedure this results in branches of *infinite* length. In fact in can be proven that even Relevant propositional logic is, therefore, *undecidable*! It can be really hard to ascertain what holds in system **R**.

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Relevance of **B** ... **R**

- The systems that extend **B** up to **R** are indeed Relevant in the sense that in a logically true conditional the *variable sharing requirement* is met.
- This can be shown (using some strange 8-valued semantics) for **R**. Given that it holds for the strongest of these systems, it has to hold for all of them.
  [cf. (Priest 2001), pp.197-98, and p.209 for a much simpler proof for logic **B.**]
- [Note that taking some strange semantics – that is not the intended semantics of our system – to prove this fact is admissible: First one proves that **R** is sound with respect to the strange semantics; then one shows that given some such semantics Irrelevant conditionals are not logically true; so, given the soundness of **R**, they cannot be proven; if they cannot be proven (a *syntactic* meta-property), they cannot be proven in the intended semantics either! This method of using arbitrary (non-intended) semantics to prove meta-theorems is often employed in paraconsistent metalogic.]

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Brady's Logic of Entailment

- Ross Brady proposes the following system as a Relevant logic that captures the intuitive notion of entailment. Its semantics is modelled on the idea of *containment of content*. The basic idea is that a content is connected to each sentence and the relation of entailment is that of containment of content.
- Brady's logic is important, since he considers it to be a universal logic that can be employed in any context [see Chap. 20]. He uses it for his version of paraconsistent set theory [see Chap. 11].
- Brady's logic, called $\mathbf{DJ^{d}Q}$**,** is very weak and has to assume axioms for relations derivable otherwise.
- Brady's logic does *neither* use the ternary accessibility relation *nor* the Routley star *. That, too, makes it worth looking at.

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DJ^{d}Q – Axioms

- Axiom schemes:

(A1) $A \to A$

(A2) $A \wedge B \to A$

(A3) $A \wedge B \to B$

(A4) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A5) $A \to A \vee B$

(A6) $B \to A \vee B$

(A7) $(A \to B) \wedge (C \to B) \to (A \vee C \to B)$

(A8) $A \wedge (B \vee C) \to A \wedge B \vee A \wedge C$

(A9) $\neg\neg A \to A$

(A10) $(A \to \neg B) \to (B \to \neg A)$

(A11) $(A \to B) \wedge (B \to C) \to (A \to C)$

(A12) $(\forall x)P(x) \to P(\text{á})$

(A13) $(\forall x)(A \to P(x)) \to (A \to (\forall x)P(x))$ *

(A14) $(\forall x)(A \vee P(x)) \to (A \vee (\forall x)P(x))$ *

(A16) $P(\text{á}) \to (\exists x)P(x)$

(A17) $(\forall x)(P(x) \to A) \to ((\exists x)P(x) \to A)$ *

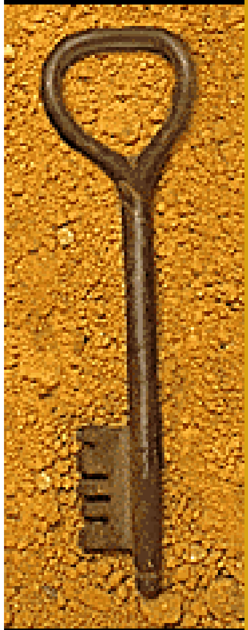
(A18) $A \wedge (\exists x)P(x) \to (\exists x)(P(x) \wedge A)$ *

* [x not free in A]

DJ^dQ – Rules

- Rules:
  - (R1) $A \rightarrow B, A \Rightarrow B$
  - (R2) $A, B \Rightarrow A \wedge B$
  - (R3) $A \rightarrow B, C \rightarrow D \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow D)$
  - (R4) $A \Rightarrow (\forall x)A$
- Meta-Rules
  - (MR1) If $A \Rightarrow B$ then also $A \vee C \Rightarrow B \vee C$
  - (MR2) If $A \Rightarrow B$ then $(\exists x)A \Rightarrow (\exists x)B$

where in both meta-rules in the derivation $A \Rightarrow B$ (R4) does not generalize on a free variable in A.



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DJ^{d}Q – Semantics

- We start with contents: The *content* of a set of sentences Γ are all the sentences that can be *analytically established* from the terms occuring in the sentences in Γ. Often we take Γ to be a singleton {A}.
- The *range* of Γ is the set of all sentences that analytically ensure that at least one sentence in Γ is true.
- Each sentence A corresponds a set of sentences c(A), which is its content (the analytic closure of A). The content set of A can be taken as a great conjunction. $A \models B$ for $B \in c(A)$ is analytic.
- Each sentence corresponds a set of sentences r(A), which is its range. The range set of A can be taken as a great disjunction.
- The logical operations can be semantically modelled by algebraic operations on these (content) sets. Every basic sentence is assigned some content. The content of $A \wedge B$ is the set of analytic consequences of the union of c(A) and c(B). The content of $A \vee B$ is $c(A) \cap c(B)$.
- Entailment is now semantical defined as *containment of content*:

  $$v(A \rightarrow B) = 1 \text{ iff } c(B) \subseteq c(A).$$

- To ensure a Relevant logic postulates for content relations are laid down, e.g. $c(A \vee B) = c(A) \cap c(B) \subseteq c(A)$.

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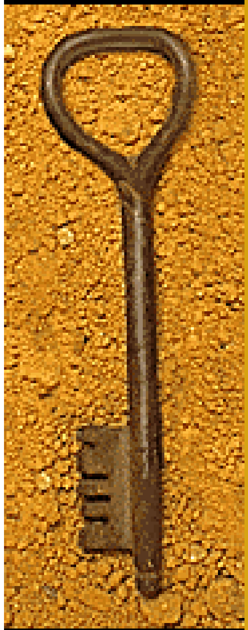
DJ^{d}Q – Semantics (II)

- An exception to this algebra is negation, since taking negation as set complement would give A ∧¬A universal content, and would validate Explosion and *verum ex qoudlibet* for entailment. The negation of a sentence can be better understood as all those sentences that represent alternative situations, i.e. the content of ¬A are all facts the absence of which made A possible: $c(\neg A) := \{B | \neg B \in r(A)\}$. This set is c(A)*. The interpretation *v* assigns c(A)* to ¬A.
- The semantics of entailment – taking it as content containment – seems to be close to our natural understanding of entailment. Nothing about accessing the accessibility of worlds is involved.
- This semantics – obviously – *presupposes* a theory of establishing the analytic consequences of a given sentence, i.e a theory of analyticity! This may require meaning postulates, which in turn may contain expressions (like "→") the semantics of which operates on contents! A meaning postulate – usually a conditional – may look like the prototypical entailment. Basic content assignment (at the beginning of the interpretation semantics) may not be so basic at all.



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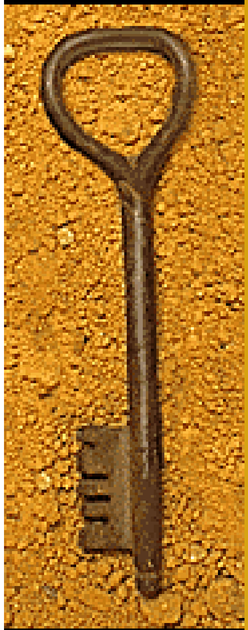
DJ^{d}Q – Metalogic

- Sentences with identical contents can always be *substituted* for each other. (For contained meaning semi-substitutivity in a context holds.)
- **DJ^d** is sound and complete relative to the semantics of contents. [cf. (Brady 1996)].
- Several typical theorems do not hold in **DJ^{d}Q**:
  Contraction, the Law of Excluded Middle [$A \vee \neg A$], Disjunctive Syllogism, the *Modus Ponens* Theorem, nor $(A \rightarrow \neg A) \rightarrow \neg A$.
- Therefore **DJ^{d}Q** satisfies the Strong Anti-Triviality Condition.



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Assessment

- The established systems of Relevant logic capture at least parts of our concepts of relevance and entailment.
- The most pressing problem is to give an intuitively convincing semantics for entailment and negation within the RL context. The Routley star is a cross violation of the Extensionality condition. Semantic work seems to be still going on; the focus on information theory may prove to be useful.
- We cannot take some RL to be our target logic for paraconsistency if it fails on the Strong Anti-Triviality Condition. Neither can be take a logic that validates Contraposition in general (as we will see in Chap. 16). The typical systems **B**, **R**, **E** are, therefore, not available to us.
- For an assessment of the information (flow) semantics consider questions (Q1), (Q2), and remember the note on situations vs. worlds.
- One of the most promising of the logics presented here is $\mathbf{DJ^{d}Q}$. It satisfies the Strong Anti-Triviality condition. It has Contraposition, but this depends on a corresponding *postulate* on content semantics, that might be dropped.

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Assessment (II)

- Another – also major – problem with Relevant Logic from a philosophical perspective is that it treats *ex contradictione quodlibet* and *verum ex quodlibet* as being on a par.
- This is not clear at all: *ex contradictione quodlibet* seems (a) intuitively implausible [we just don't reason that way] and (b) would be disastrous if valid. Both points do not hold that obviously for *verum ex quodlibet*: certainly it would not be disastrous if valid, contra (b); and it need not be intuitively implausible, we just don't care to give more reasons for what is already established as valid, contra (a).
- Many of the more contrived semantic manoeuvres in Relevant Logic are tied to avoiding some form of *verum ex quodlibet sequitur*. We can avoid all these troubles simply accepting *verum ex quodlibet*, as many paraconsistent logics do.

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Questions

- (Q1) Consider the information flow characterization of the ternary accessibility relation. Given that it is an appropriate characterization of accessibility relations within distributed systems (information channels) does it *by this* provide an analysis/model for our concept of *semantic* entailment?
- (Q2) Consider the characterization of the Routley star in terms of compatible (completed) situations. Given that this is a good characterization of negation, is it truth-functional or intensional or neither of them? How to we ascertain the truth of $\neg A$? If every situation was compatible with itself, what would this imply for Explosion?
- (Q3) Why is the content of $A \vee B$ less than the content of the single sentences: $c(A) \cap c(B)$ (i.e. the cut and not the union $c(A) \cup c(B)$)?

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Exercises

- (Ex1) Using the truth condition (R→) show that *verum ex qoudlibet sequitur*, A→(B→A), is not a logical truth.
- (Ex2) Show that given the restriction on R associated with Contraction (A14) cannot be invalidated.
- (Ex3) Why is the *Modus Ponens* theorem not valid in the content semantics of $\mathbf{DJ^dQ}$? Consider the *closure* conditions of contents (what do they involve and what do they not involve), and the *content* of the conjunction A ∧ (A → B).

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Further Reading

- On Relevant Logics in general you may look in the two volumes of *Entailment* (ed. by A. Anderson et al.) or the two volumes of *Relevant Logics and their Rivals* (ed. by R. Routley, R. Brady et al.), but for a introduction to these logics you better use: Read, Steven. *Relevant Logic*. Oxford, 1994. A short introduction you find in (Priest 2000). A philosophical introduction and discussion you find in: Mares, Edwin. *Relevant Logic*. A Philosophical Interpretation. Cambridge, 2004.
- The idea of a information (flow) semantics for Relevant logics you can find in: Mares, Edwin. "Relevant Logic and the Theory of Information", *Synthese*, 109 (1996), pp. 345-60; and: Restall, Greg. "Information Flow and Relevant Logics", in: Seligmann, Jerry/ Westerstahl, Dag (Eds). *Logic, Language and Computation*. Stanford, 1996, pp. 463-77. On theory of information flow see (Barwise/ Seligman 1997); on situation semantics (Devlin 1991).
- Brady's logic is presented in: "Entailment, Negation and Paradox Solution", in: (Batens et al. 2000), pp.113-36.
- An approch similar to Relevant logic in requiring the sharing of content is the relatedness logic in (Epstein 1995), but it's explosive.