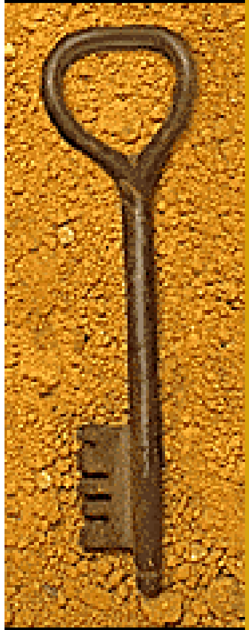


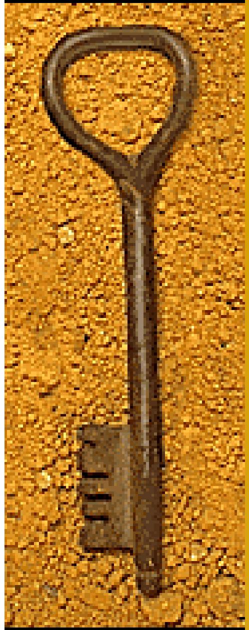
$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

A Sequential Calculus

- In this chapter we look at another calculus that fits the criteria of Chapter 3, including the Strong Anti-Triviality Condition.
- This calculus contains a conditional connective that is stronger than the material conditional and satisfies *Modus Ponens* and *Transitivity*.
- The logic is not, however, a Relevant Logic in the sense of Chapter 5. The semantics of the entailment connective does not employ the ternary accessibility relation or some variant of it.
- The logic **SKP** is based on (Priest 1980), supplemented with standard quantificational theory; the revisions (like dropping Contraposition and restricting Substitution of Identicals) will be motivated in Chapter 16.



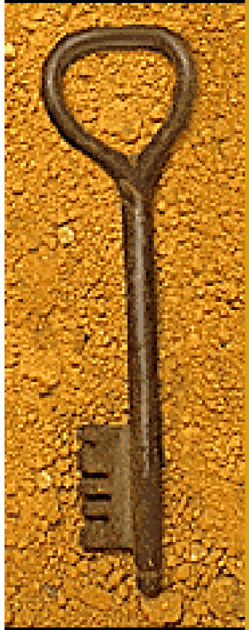
p ∧¬p

Manuel Bremer
Centre for Logic,
Language and
Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

Sequential Calculi

- A sequential calculus gives a logic of sequences (of well-formed formula). The formulas in the sequence are separated by the symbol "||-" which means that the formulas on the right of it, taken conjunctively make true the formulas to the left of it, taken disjunctiveley. Formulas on the right and left of "||-" are separated by commas.
- The original approach (Gentzen 1934/35) employed rules for the introduction/elimination of connectives and quantifiers on the right and the left side. In our calculus we have only single formulas on the right side (the case of a disjunction of one disjunct only). The calculus is not developed in the systematic fashion of introduction and elimination rules (like in Natural Deduction systems descending from sequential systems), but by giving axioms and rules according to our needs and in the not so systematic fashion that results from restricting standard logic.



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

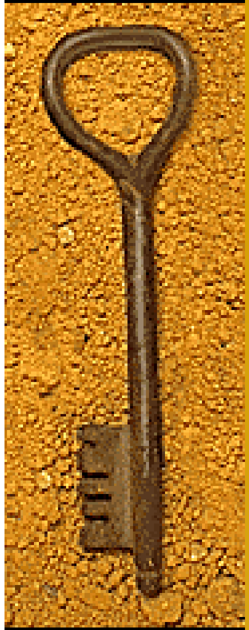
HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

||-

- A sequence is a formula with exactly one "||-" occuring in it. On the left of "||-" are nil, one or more than one formula separated by commas, taken conjunctively. On the left of "||-" stands a single formula.
- These lines like

  A, B ||-C

  are *sentences of the language*, not the metalanguage. "||-" expresses within the language the inheritance of truth from the formulas on the left to the formula on the right. In fact it works thus like "⊃". One may introduce "⊃" into the language of **SKP** using its usual definition, but, of course, many of its theorems and consequences are missing.
- **SKP** works with the distinction between inheritance of truth, symbolized by "||-" and entailment, symbolized by " →".
- There are no rules for "||-" itself.
- Rules of a sequential calculus tell us how to derive a new *sequence* given other sequences. Derivation *only* contain lines with sequences.
- In the example A, C and B do not contain occurences of "||-".



p ∧¬p

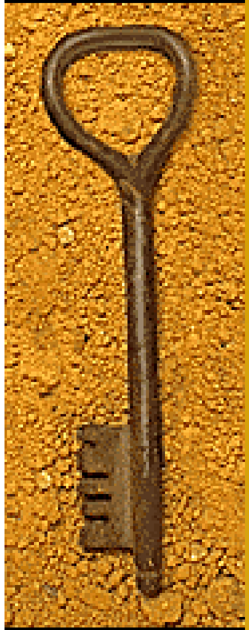
Manuel Bremer
Centre for Logic, Language and Information

||- Semantics

- How "||-" works can also be seen in its semantics. Since it is part of our language we need semantic clauses for "||-"; especially in evaluating lines in deductions.
- The only case in which inheritance of truth is false is that in case of a true left hand side the right side is false *only*. In case the right hand side is antinomic truths is – trivially – inherited. Since *simply false* is binary [see Chap. 4], a *sequence* can be only true or false, never both!
- Writing thus $(A_1 \ldots A_n) \|{-} B$ for $v((A_1 \ldots A_n) \|{-} B, 1)$ and $\neg((A_1 \ldots A_n) \|{-} B)$ for $v((A_1 \ldots A_n) \|{-} B, 0)$ we state the truth conditions:

(S||-) (i) $(A_1 \ldots A_n) \|{-} B \Rightarrow \neg(\exists v)(v(A_1,1) \wedge \ldots v(A_n,1) \wedge \neg\, v(B,1))$
(ii) $\neg(\exists v)(v(A_1,1) \wedge \ldots v(A_n,1) \wedge \neg\, v(B,1)) \Rightarrow (A_1 \ldots A_n) \|{-} B$
(iii) $(\exists v)(v(A_1,1) \wedge \ldots v(A_n,1) \wedge \neg\, v(B,1)) \Rightarrow \neg((A_1 \ldots A_n) \|{-} B)$
(iv) $\neg((A_1 \ldots A_n) \|{-} B) \Rightarrow (\exists v)(v(A_1,1) \wedge \ldots v(A_n,1) \wedge \neg\, v(B,1))$

- (iii) and (iv) give the contrapositives of (i) and (ii).



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

||- and Natural Deduction

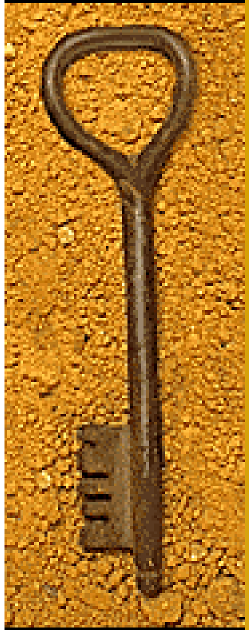
- We have used some natural deduction systems already and later will introduce some of them more systematically. Sequent systems are the historic ancestors of these systems.
- For a better understanding of "||-" it might help you to see the line

  $$\text{n.} \qquad (A_1 \ldots A_n) \; ||\!\!- B$$

  in a deduction in this sequent calculus as another representation of what in our natural deduction format would look like:

  $$\text{n.} \langle A_1 \ldots A_n \rangle \qquad B$$

  "||-" would thus express the presense of premises in the object language itself, what is *shown* only by the way of rendering natural deduction proofs.

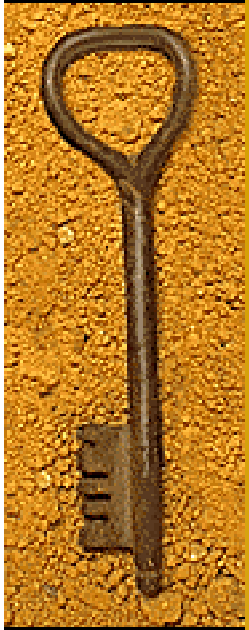


p ∧¬p

Manuel Bremer
Centre for Logic,
Language and
Information

Syntax

- The language of **SKP** is the same as that of the Relevant Logics, supplemented by the symbol "||-". The formation rules are adapted accordingly for sequences and the definition of a derivation.
- We take conjunction, negation, and entailment "→" as the basic connectives and define in the usual way:
- (D1) $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$
- (D2) $A \vee B := \neg(\neg A \wedge \neg B)$
- One may add the definitions:
  (D3) $A \supset B := \neg A \vee B$
  (D4) $A \equiv B := (A \supset B) \wedge (B \supset A)$
  (D5) $\top := \neg A \vee A$
- The quantifies are both introduced in the quantificational rules.
- ||-B says that the premise set from which B inherits truth is empty, i.e. B is considered to be logically true. [A, B are again schematic letters of the meta-language for object-language formulas "p", "p → q" etc. (cf. Chap. 1)]
- In stating rules, executing set theoretical operations on the premise sets we use "Γ", "Π" and "Θ" for premise *sets*. These sets may be empty or may be identical.



$p \wedge \neg p$

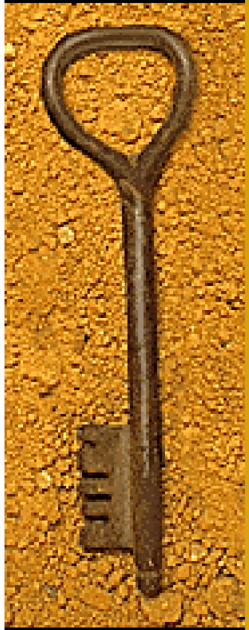
Manuel Bremer
Centre for Logic, Language and Information

SKP Axioms

- The axiom schemes of **SKP** are:

(A1) $\{A\} \Vdash A$
(A2) $\Vdash A \wedge B \rightarrow A$
$\Vdash A \wedge B \rightarrow B$
(A3) $\Vdash A \rightarrow A \vee B$
$\Vdash B \rightarrow A \vee B$
(A4) $\Vdash A \rightarrow \neg\neg A$
(A5) $\Vdash \neg\neg A \rightarrow A$
(A6) $\Vdash A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
(A7) $\Vdash A \vee \neg A$
(A8) $\Vdash (\forall x)x=x$

- Everything makes itself true (Axiom 1). Corresponding to the Extensionality Condition conjunction and disjunction are duals and have their usual properties, including distribution (Axioms 2, 3, 6). Negation is treated classically, not intuitionistically, validating *Tertium Non Datur* (Axioms 4, 5, 7). The last axiom is the basic axiom for **SKP** being a quantificational logic with identity.
- The real power of **SKP** resides with the rules.



p ∧¬p

Manuel Bremer
Centre for Logic,
Language and
Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

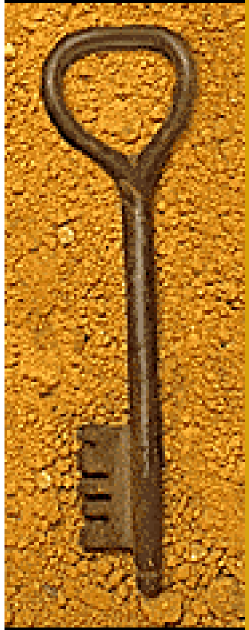
SKP Propositional Rules

- The rules of **SKP** are:

(R1) $\Gamma \cup \{A\} \Vdash B \;\&\; \Pi \cup \{C\} \Vdash B \Rightarrow \Pi \cup \Gamma \cup \{A \vee C\} \Vdash B$
(R2) $\Gamma \Vdash (B \to A) \;\&\; \Pi \Vdash (B \to C) \Rightarrow \Pi \cup \Gamma \Vdash (B \to A \wedge C)$
(R3) $\Gamma \Vdash (A \to B) \;\&\; \Pi \Vdash (C \to B) \Rightarrow \Pi \cup \Gamma \Vdash (A \vee C \to B)$
(R4) $\Gamma \Vdash (A \to B) \;\&\; \Pi \Vdash A \Rightarrow \Pi \cup \Gamma \Vdash B$
(R5) $\Gamma \Vdash (A \to B) \;\&\; \Pi \Vdash (B \to C) \Rightarrow \Pi \cup \Gamma \Vdash (A \to C)$
(R6) $\Gamma \Vdash A \;\&\; \Pi \Vdash C \Rightarrow \Pi \cup \Gamma \Vdash (A \wedge C)$

- Rules (R1) and (R3) corresponds to prefixing, (R2) to postfixing. (R6) states conjunction introduction. (R4) is *Modus Ponens* for entailment. (R5) is transitivity for entailment. **SKP**, thus, meets the *Modus Ponens* Condition, and seems to meet the Minimal Damage Condition with respect to the conditional connective. Substitution of (semantic) equivalents holds by

(R7) $\Gamma \Vdash (A \leftrightarrow B) \Rightarrow \Gamma \Vdash (C \leftrightarrow C(A/B))$
where C(A/B) is built by replacing all occurences of A in C by B.



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

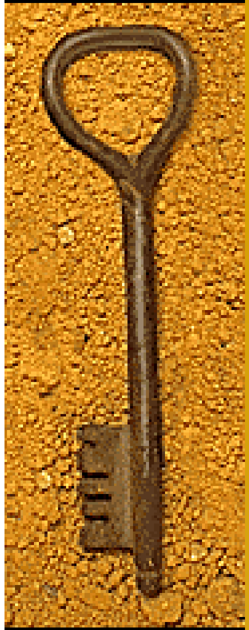
SKP Propositional Rules (II)

- **SKP** contains also – in addition to transitivity of entailment, and like other sequential calculi – a Cut rule for "||-":

$$(\text{R8})\ \Gamma \cup \{A\} \Vdash B \ \&\ \Pi \Vdash A \ \Rightarrow \Pi \cup \Gamma \Vdash B$$

- Contraposition can be *added* (to give us **SKP**$^{+}$):

$$(\text{R14})\qquad \Gamma \Vdash (A \rightarrow B) \Rightarrow \Gamma \Vdash (\neg B \rightarrow \neg A)$$



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

SKP Quantificational Rules

- **SKP** contains standard quantificational rules.
  [Remember: "P( )", "á" ... are schemata for predicates and constants.]

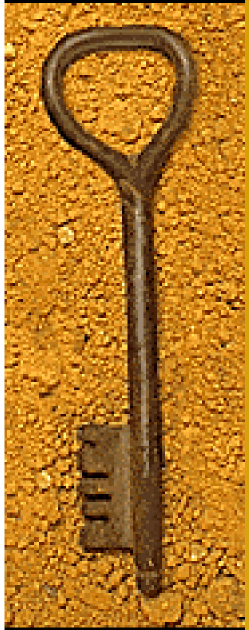
(R9) $\Pi\|\!-P(\acute{e}) \Rightarrow \Pi\|\!-(\exists x)P(x)$

(R10) $\Pi\|\!-(\exists x)P(x)\ \&\ \{P(\acute{a})\} \cup \Gamma\|\!-B \Rightarrow \Pi \cup \Gamma\ \|\!-B$
where á does not occur in B, $(\exists x)P(x)$ nor in any premise in $\Pi \cup \Gamma$.

(R11) $\Pi\|\!-P(\acute{e}) \Rightarrow \Pi\|\!-(\forall x)P(x)$
where é does not occur in any premise in Π.

(R12) $\Pi\|\!-(\forall x)P(x) \Rightarrow \Pi\|\!-P(\acute{a})$

SKP Quantificational Rules (II)



p ∧¬p

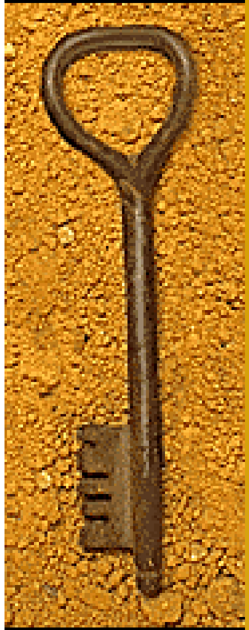
Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

- In case of Substitution of Identicals we introduce the set Ξ of inconsistent objects (i.e. objects such that: $P_i(\acute{a}) \wedge \neg P_i(\acute{a})$ for some P_i). Let δ be the denotation function of quantificational semantics. Then:

  $$(R13)\ \delta(\acute{a}) \notin \Xi \ \&\ \Gamma \Vdash \acute{a}=\acute{e} \ \&\ \Pi \Vdash P(\acute{e}) \Rightarrow \Pi \cup \Gamma \Vdash P(\acute{a})$$

  If á does not denote an inconsistent object then substitution applies.
- Applying (R13) is problematic, since to apply it we have to know whether $\delta(\acute{a}) \notin \Xi$. If we want to express the consistency of an object *in* our language we needed set theoretical expressions. For the purposes of Chap. 16 the restriction $\delta(\acute{a}) \neq m$ suffices with m being the least inconsistent number. Nevertheless this would be needed as additional premise. (Remember that in $\mathbf{LPQ}^{=}$ substitution of identicals fails *in general*.)
- (R13) without the restriction concerning inconsistent objects can – with (R14) – be taken as rule of $\mathbf{SKP}^{+}$.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Derivations

- A derivation is a list of numbered lines, each containing a sequence. On the right we note how the line was generated.
- For better readability we number the premises and refer to them in a sequence by writing down the number, for example:

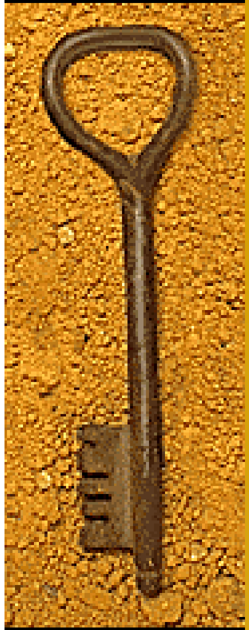
$$\vdash_{SKP}((\forall x)F(x) \| \!- F(a))$$

Proof:

(1) $(\forall x)F(x)$

1. $1 \| \!- (\forall x)F(x)$	A1
2. $1 \| \!- F(a)$	R12, 1 ■

- Do not confuse "$\vdash_{SKP}$" with "$\| \!-$"!
- Numbers on the right of course refer to lines of the derivation to which the rule in question has been applied (*not* to formulas to the left of the "$\| \!-$").



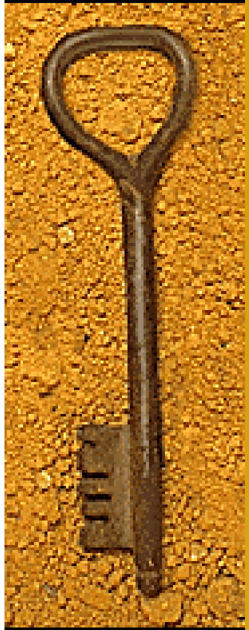
p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

SKP Theorems

- (SKPT2) $(\forall x)(F(x)\rightarrow G(x)), (\forall x)F(x) \Vdash (\forall x)G(x)$
  Proof (1) $(\forall x)(F(x)\rightarrow G(x))$
  (2) $(\forall x)F(x)$
  1. $1 \Vdash (\forall x)(F(x)\rightarrow G(x))$ A1
  2. $2 \Vdash (\forall x)F(x)$ A1
  3. $1 \Vdash (F(a)\rightarrow G(a))$ R12, 1
  4. $2 \Vdash F(a)$ R12, 2
  5. $1,2 \Vdash G(a)$ R4, 3, 4
  6. $1, 2 \Vdash (\forall x)G(x)$ R11, 6 ■

  Note that the corresponding consequence relation was invalid in **LPQ**.
- (SKPT3) $(\forall x)(F(x) \wedge G(x)) \Vdash (\forall x)F(x) \wedge (\forall x)G(x)$
  Proof (1) $(\forall x)(F(x) \wedge G(x))$
  1. $1 \Vdash (\forall x)(F(x) \wedge G(x))$ A1
  2. $1 \Vdash F(a) \wedge G(a)$ R12, 1
  3. $\Vdash F(a) \wedge G(a) \rightarrow F(a)$ A2
  4. $1 \Vdash F(a)$ R4, 3, 4
  5. $1 \Vdash (\forall x)(F(x)$ R11, 4
  6. $\Vdash F(a) \wedge G(a) \rightarrow G(a)$ A2



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

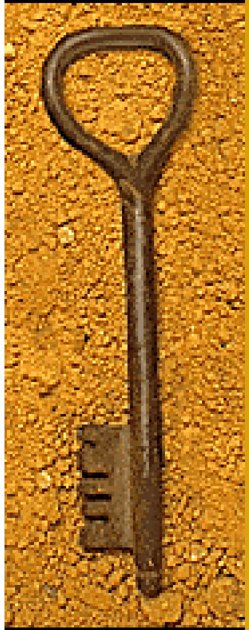
SKP Theorems (II)

7. $1 \Vdash G(a)$	R4, 2, 6
8. $1 \Vdash (\forall x)G(x)$	R11, 7
9. $1 \Vdash (\forall x)F(x) \wedge (\forall x)G(x)$	R6, 5, 7 ■

- (SKPT4) $\Vdash p \leftrightarrow \neg\neg p$
  Proof
  by (A4), (A5), (D1), (R6) ■
- (SKPT5) $\Vdash p \wedge p \leftrightarrow p$
  Proof

1. $\Vdash p \wedge p \rightarrow p$	A2
2. $\Vdash p \rightarrow \neg\neg p$	A4
3. $\Vdash p \leftrightarrow \neg\neg p$	Theorem 3
4. $\Vdash p \rightarrow p$	R7, 2, 3 [=(SKPT6)]
5. $\Vdash p \rightarrow p$	R7, 2, 3
6. $\Vdash p \rightarrow p \wedge p$	R2, 4, 5
7. $\Vdash p \wedge p \leftrightarrow p$	D1, 1, 6 ■

- (SKPT7) $\Vdash p \vee p \leftrightarrow p$
  Proof
  by (A3), (SKPT6), (R3), (D1) ■



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

SKP Theorems (III)

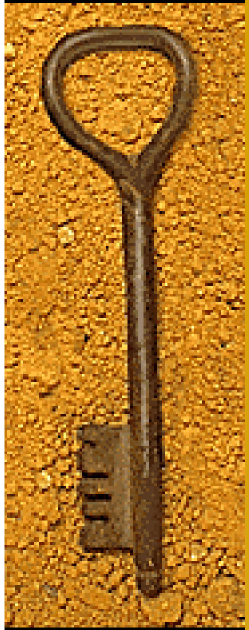
- (SKPT8) $\|\!-(p \wedge q) \vee (p \wedge r) \rightarrow p \wedge (q \vee r)$
  Proof

1. $\|\!-(p \wedge q) \rightarrow q$	A2
2. $\|\!-q \rightarrow q \vee r$	A3
3. $\|\!-(p \wedge q) \rightarrow (q \vee r)$	R5, 1, 2 (Transitivity)
4. $\|\!-(p \wedge q) \rightarrow p$	A2
5. $\|\!-(p \wedge q) \rightarrow p \wedge (q \vee r)$	R2, 3, 4
6. $\|\!-(p \wedge r) \rightarrow p \wedge (q \vee r)$	(similar to lines 1-5)
7. $\|\!-(p \wedge q) \vee (p \wedge r) \rightarrow p \wedge (q \vee r)$	R3, 5, 6 ■

- Interestingly enough the Law of Non-Contradiction holds in **SKP**:
  (SKPT9) $\|\!- \neg(p \wedge \neg p)$
  Proof

1. $\|\!- (p \vee \neg p)$	A7
2. $\|\!- \neg(\neg p \wedge \neg \neg p)$	D2, 1
3. $\|\!- p \leftrightarrow \neg \neg p$	Theorem 3
4. $\|\!- \neg(p \wedge \neg p)$	R7, 2, 3 ■

- (SKPT10) $(p \rightarrow (q \rightarrow r)), p \wedge q \|\!-r$ [like Importation]
  Proof
  by (A1), (A2), (R4) ■



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

SKP Theorems (IV)

- Disjunctive and Conjunctive Syllogism are not derivable in **SKP**.
- Indirect methods of proofs are, because of the absence of Contraposition, not available in **SKP**, but in **SKP**$^+$ (see Exercises).
- There are two modest ways of something like indirect argumentation in **SKP**, namely:

(DR1) $A \Vdash (\neg A) \Rightarrow \Vdash (\neg A)$

Proof

1. $A \Vdash (\neg A)$	Assumption (using schematic A)
2. $\neg A \Vdash (\neg A)$	A1
3. $A \vee \neg A \Vdash (\neg A)$	R1, 1, 2
4. $\varnothing \Vdash A \vee \neg A$	A7
5. $\varnothing \Vdash \neg A$	R8, 3, 4 ■

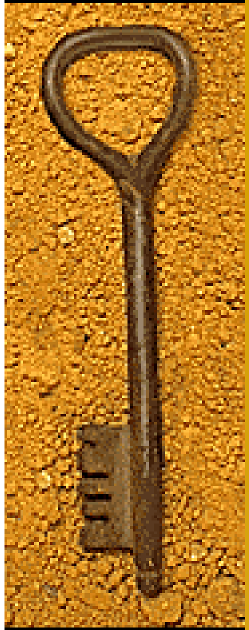
(DR2) $\Vdash (A \rightarrow \neg A) \Rightarrow \Vdash (\neg A)$

Proof

by (SKPT6), (R3), (A7), (R4) ■

SKP Theorems (V)

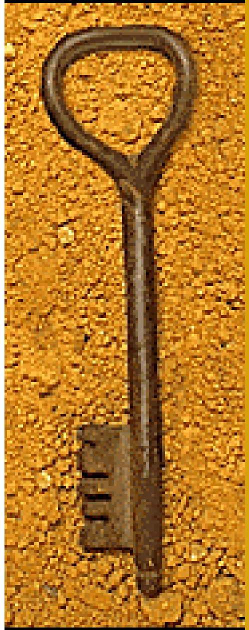
- The dualities of the extensional "⊃" have, of course, no corresponding dualities involving entailment, "→". Nevertheless there are some theorems [see (SKPT10) and Exercise (2)] with similar content.
- Nested entailments might be problematic with respect to our natural understanding of entailment (a nested entailment would say something like that the content of the antecedent contains that some other content containment obtains). The rules of **SKP** do not allow to introduce some sentence into the antecedent of "→" crossing " ||- ".



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF



p ∧¬p

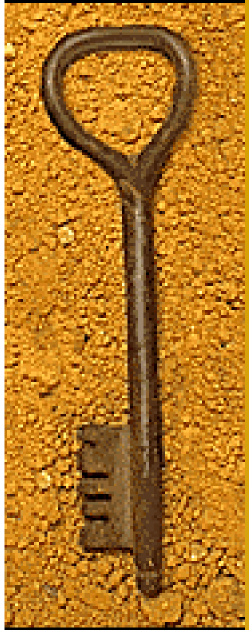
Manuel Bremer
Centre for Logic, Language and Information

SKP Semantics

- **SKP** can be given a semantics that combines elements of a possible worlds semantics (for entailment) with a special role for the actual world @, elements of extensional truth conditions for the extensional connectives, and the quantificational semantics we introduced in Chap. 4, extended to the modal case.
- A **SKP** model *M* is a tuple <W, R, @, D_i, *v*, G> where W is a set of possible worlds, R is the accessibility relation as it obtains in the model (there may be some general requirements on R), @ is the actual world, D_i is the set of domains (one domain for each possible world), *v* is the interpretation relation, G is the set of variable assignments.
- The interpretation relation *v* has a further argument place now for a possible world: each sentence is evaluated relative to a world; each basic predicate gets its extension and anti-extension relative to a world. We take individual constants to be *rigid designators*.
- We require for accessibility that all worlds are accessible from the actual world – one may say, because there are alternatives *for us*:
  (SKPSR1) $(\forall w \in W)R(@,w)$
- We already have seen how to evaluate the " ||- " in a sentence.

SKP Semantics (II)

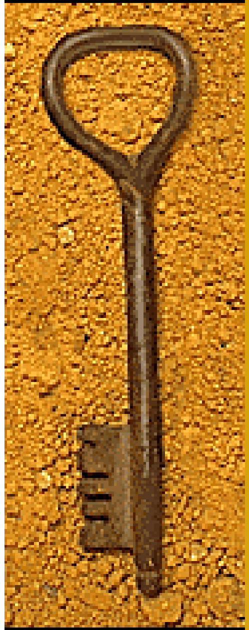
- To deal with identity we require any interpretation to treat the identity symbol as a constant referring to the identity relation on the domain:
  (SKP=) $(\forall v)(v("=")=id(D_i))$
  One might allow an anti-extension for "=", but these would be objects that aren't identical to themselves! We don't do this and keep (A8).
- The standard truth conditions for the extensional connectives and the quantifiers can be used [see Chap. 4], the only intensional connective is entailment.
- Intuitively an entailment is true in a world iff in all worlds which are accessible the truth of the antecedent brings with it the truth of the consequent; an entailment is false in a world iff a world is accessible in which the antecedent is true, but the consequent isn't.
  [Note that we don't have in **SKP** the contrapositive condition.]
- To give a formal statement of the intuitive truth condition one has to state *all cases* of the equivalence relation, and keep an eye to the distinction between false and *simply* false entailments. Furthermore we don't use a conditional " ⇒ " in the meta-language that allows for Contraposition, since the object- *vs.* meta-language distinction is ultimately to be dropped. This leads to the complex condition:



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF



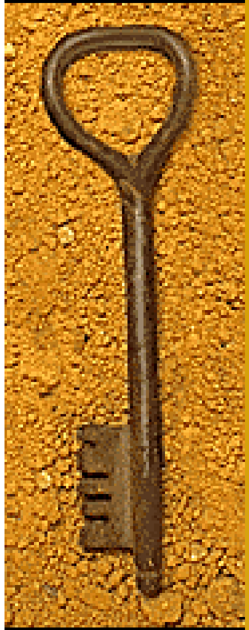
$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

SKP "→" Semantics

(SKPS→)

- (i) $v(w, A \rightarrow B, 1) \Rightarrow (\forall w' \in W)(R(w,w') \wedge v(w',A,1) \Rightarrow v(w',B,1))$
- (ii) $\neg\, v(w, A \rightarrow B, 1) \Rightarrow \neg(\forall w' \in W)(R(w,w') \wedge v(w',A,1) \Rightarrow v(w',B,1))$
- (iii) $(\forall w' \in W)(R(w,w') \wedge v(w',A,1) \Rightarrow v(w',B,1)) \Rightarrow v(w, A \rightarrow B, 1)$
- (iv) $\neg(\forall w' \in W)(R(w,w') \wedge v(w',A,1) \Rightarrow v(w',B,1)) \Rightarrow \neg v(w, A \rightarrow B, 1)$
- (v) $v(w, A \rightarrow B, 0) \Rightarrow (\exists w' \in W)(R(w,w') \wedge v(w',A,1) \wedge v(w',B,0))$
- (vi) $\neg\, v(w, A \rightarrow B, 0) \Rightarrow \neg(\exists w' \in W)(R(w,w') \wedge v(w',A,1) \wedge v(w',B,0))$
- (vii) $(\exists w' \in W)(R(w,w') \wedge v(w',A,1) \wedge v(w',B,0)) \Rightarrow v(w, A \rightarrow B, 0)$
- (viii) $\neg(\exists w' \in W)(R(w,w') \wedge v(w',A,1) \wedge v(w',B,0)) \Rightarrow \neg v(w, A \rightarrow B, 0)$



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

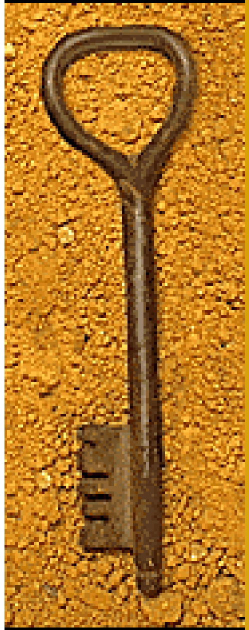
HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

SKP Validity

- Truth *in a model* is bound to being true *in the actual world* (of the model). All others worlds are only tools of evaluating formula.
- *Validity* is truth in all models (i.e. in all of their *actual* worlds):

  (SKPV) $\Gamma \models A \Leftrightarrow (\forall M' \in M)((\forall B \in \Gamma)\ v'(@,B,1) \Rightarrow v'(@,A,1))$

  A consequence being valid in case that for all models the conclusion is (at least) true at the actual world of the model if the premises are (at least) true at the actual world of that model. A valid sentence is one that follows from the empty set of premises.
  [Note: Do not confuse " |= " with " ||- "!]
- On the one hand this definition of validity is non-standard in as much as standard modal logic require a logical truth to be true in all worlds of all models. On the other hand *we* are in the actual world (or the thing corresponding to that description) and all other possible worlds are only tools in our reasoning what sentences are true. Thinking of some of *them* that they might be actual is thinking of *another model*. So it is not clear that the notion of validity of **SKP** fares worse than the standard notion.



$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

Invalid Formula

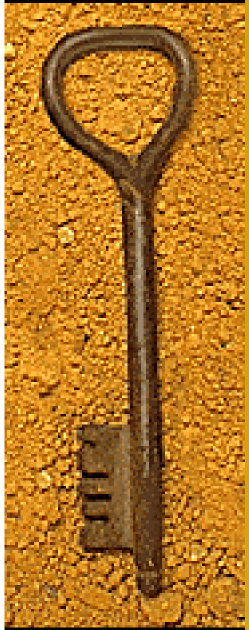
- All worlds being accessible from the actual world makes the actual world accessible from itself, otherwise R need not be reflexive.
- We depict accessibility by arrows "→ "and write true sentences beneath a given world letter.
- Given this semantics we can show that **SKP** fulfills the Strong Anti-Triviality Condition. We show that Contraction/Absorption is not valid in this semantics, neither is the *Modus Ponens* Theorem.
- (SKPNT1) $\varnothing \|\!\!- p\rightarrow(p\rightarrow q) \ |\!\neq_{\mathbf{SKP}} \varnothing \|\!\!-(p\rightarrow q)$

  Proof

  Let @ and w'' be reflexive. w only sees w''. w'' only sees w''.

	@	→	w	→	w''
1.			p		p
2.					q
3.			$p\rightarrow q$		$p\rightarrow q$
4.	$p\rightarrow(p\rightarrow q)$				

  [The third line occurs, since w only sees w'', and w'' only sees itself, but @ sees w, so that $p\rightarrow q$ does not hold at @. $p\rightarrow(p\rightarrow q)$ does however hold at @, since @ sees w and w''. So the premise of Contraction is true at the actual world, but the consequent is not. So Contraction fails.]



p ∧¬p

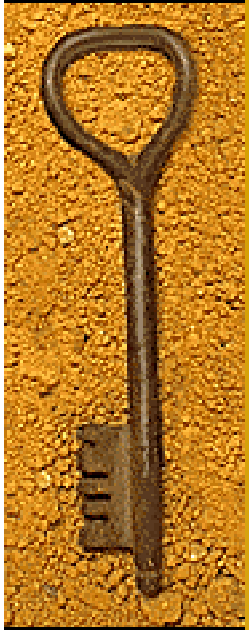
Manuel Bremer
Centre for Logic, Language and Information

Invalid Formula (II)

- (SKPNT2) $\varnothing \not\models_{SKP} p \wedge (p \rightarrow q) \rightarrow q$
  Proof

	@	← →	w
1.	p		p
2.	q		
3.			$p \rightarrow q$
4.			$p \wedge (p \rightarrow q)$

- Line 3 occurs, since w sees @ only (not itself). Line 4 is truth functional from other sentences in w. @, however, does not have "$p \rightarrow q$", since @ sees w, and so does not have "$p \wedge (p \rightarrow q)$". @ sees a world in which the antecedent of the *Modus Ponens* Theorem is true, but the consequent is not, so the supposed theorem fails.
- **SKP** does further on invalidate *verum ex qoudlibet sequitur*.
- Entailment in **SKP** is *weaker* than meaning containment (by meaning postulates) in S5 modal logics, where accessibility is an equivalence relation, since R is *not that restricted* in **SKP** semantics – otherwise Contraction and *verum ex qoudlibet sequitur* would be valid!



$p \wedge \neg p$

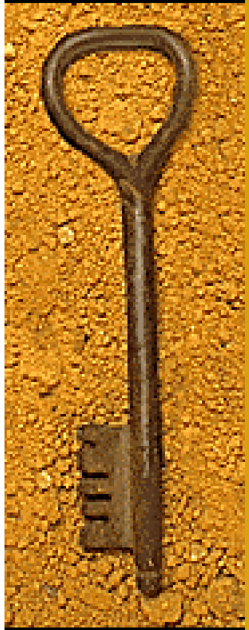
Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE UNIVERSITÄT DÜSSELDORF

SKP Meta-Logic

- **SKP** can be shown to be *sound and non-trivial* (i.e. from true sequents we are never let to sequences that are simply false by the rules of inference, and there is at least one well-formed formula that cannot be derived with that logic). Simply consistency is, of course, not to be had in a system that is (later) extended by principles that give rise to antinomies.
- (SKPMT1) $\Gamma \vdash_{SKP} A \Rightarrow \Gamma \models_{SKP} A$ (*Correctness*)

Proof (Outline)
(A1) is trivially true. (A2)-(A7) hold, since in the worlds the truth functional dependencies hold [e.g. in the case of (A2) the truth of $A \wedge B$ contains the truth of A and of B]. All rules refer only to inheritance of truth (i.e. they do not *cross* worlds). (R1), (R6), (R8) are truth functionally truth preserving. (R4) is valid, since @ is reflexive. (R6) holds, since if in every A world B holds, and in very B world C holds, then in every A world C holds. In similar vain (R2) and (R3) hold. Substitution of proven equivalents, (R7), trivially preserves truth. The quantificational rules can be justified in the same way as in case of **LPQ**. (A8) is true by (SKP=). (R13) is weaker than the usual rule.



p ∧¬p

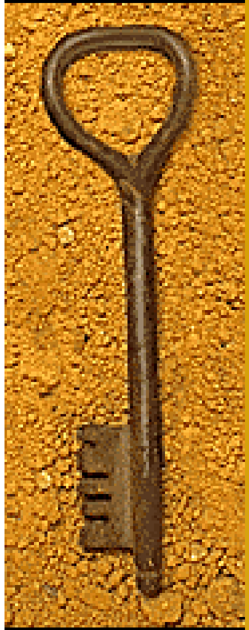
Manuel Bremer
Centre for Logic, Language and Information

SKP Meta-Logic (II)

- The theorems of **SKP** are – as we have just seen – correct with respect to its modal semantics. And we have proven – in (SKPNT1) and (SKPNT2) – that some sentences are *not valid*, and that some standard consequence relationships are *not valid*. If these theorems could be proven nevertheless, **SKP** would not be correct, but it is, so these sentences are *not provable* as well. This gives us *non-triviality*:

(SKPMT2) $(\exists A)\ \nvdash_{SKP} A$

- A completeness proof of **SKP** has not been given so far.
- With respect to the criteria given [in Chap. 3] to avoid Curry style paradoxes note also that there is a Cut rule in **SKP**, but no (derived) rule of Conditionalization.

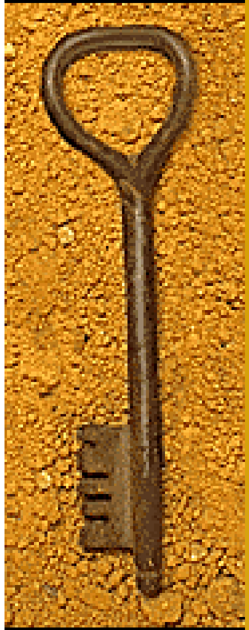


SKP Trees

- The trees for **LPQ** [see Chap. 4] may be extended to **SKP** trees. The rules for the extensional connectives and the quantifiers remain the same, we need rules for " ||- " and "→". Since "→" is a modal connective, these rules will "create" accessible worlds. So the trees have to be marked at which world a sentence is evaluated thus.
- We change the trees to:
  1. Evaluate the sentence A to be checked with ∇A in @.
  2. If a sentence contains " ||- ", then [see (S||-)]:
     (i) T(A_i ||-B) or Δ(A_i ||-B) resolves to ∇A_i *or* T(B)
     (ii) F(A_i ||-B) or ∇(A_i ||-B) resolves to ΔA_i *and* ∇B
  3. If "→" is the main connective to be resolved, then:
     (i) If w has T(A → B), then in all accessible worlds w' T(B) is the case if either T(A) or ΔA is. [see clauses (ii) and (iii) of (SKP→)]
     (ii) If w has ∇(A → B), then add an accessible w' where we have TA and ∇B. [see clauses (ii) and (iv) of (SKP→)]
     (iii) If w has F(A → B), then add an accessible w' where we have TA and FB. [see clauses (v) and (vii) of (SKP→)]
     (iv) If w has Δ(A → B), then in all accessible worlds w' ΔB is the case if either T(A) or ΔA is. [see clauses (vi) and (viii)]

p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

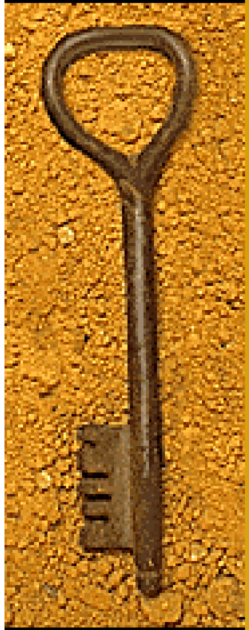
HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

SKP Trees (II)

- ...
  4. Within a world w employ the other (**LPQ**) rules.
  5. All worlds have to be accessible from @.
  6. If in one world an incompatible evaluation of a sentence occurs the sentence checked is *valid*.
- These trees provide *not* a decision procedure, but we can say:

(SKPMT3) (i) $\not\models_{SKP} A \Rightarrow$ A's tree is not closed.
(ii) A's tree closes $\Rightarrow\ \models_{SKP} A$

Proof (Outline)
In case the tree closes, the tableau, the rules of which correspond to the **SKP** semantic rules, contains incompatible evaluations; so there is not a *single* evaluation which makes A false only, so A has a designated truth value in all interpretations. If A is not valid, there will be an evaluation of the subformulas of A – following the decomposition of A like the semantic rules do – which makes A false only, so that the tree does not close.



p ∧¬p

Manuel Bremer
Centre for Logic, Language and Information

SKP Trees – Examples (I)

- As an example let us see why Contraposition fails in **SKP**:

@ $\nabla(p \rightarrow q \mathrel{||\!-} \neg q \rightarrow \neg p)$

$\downarrow$ (rule for "||-" (ii))

@ $T(p \rightarrow q), \nabla(\neg q \rightarrow \neg p)$

$\downarrow$ (added world after "→" (ii))

w $T(\neg q), \nabla\neg p$

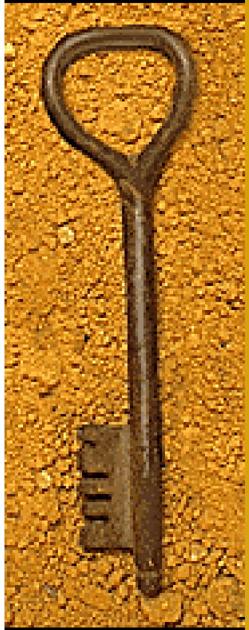
$\downarrow$ (negation rules in w)

w $Fq, \Delta p$

$\downarrow$ (rule "→" (i), @ sees w)

w Tq

The tree is not closed. Tq and Fq are compatible.



p ∧¬p

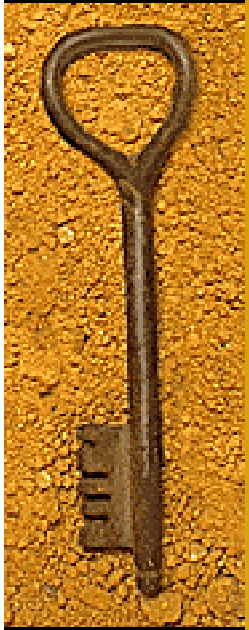
Manuel Bremer
Centre for Logic, Language and Information

SKP Trees – Examples (II)

- As an example let us see why the following formula (used in one version of a Curry Paradox [in Chap. 2]) fails in **SKP**:
  $\Vdash (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

@ $\nabla\ (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
↓ (rule for "$\nabla \rightarrow$ ", new w')
w' $T(p \rightarrow (q \rightarrow r)), \nabla((p \rightarrow q) \rightarrow (p \rightarrow r))$
↓ (rule for "$\nabla \rightarrow$ ", new w")
w" $T(p \rightarrow q), \nabla(p \rightarrow r)$
↓ (rule for "$\nabla \rightarrow$ ", new w"')
w"' $Tp, \nabla r$
↓ (rule for "$T\rightarrow$ ", in w" w.r.t. w"')
w"' $Tp, \nabla r, Tq$

The tree is not closed. Even if w' saw w"' we would only have to add $T(q \rightarrow r)$ into w"', but this is still an open tree, since decomposing $T(q \rightarrow r)$ yields requirements on worlds seen by w"', and as long as w"' does not see itself, which it need not, everything is fine.



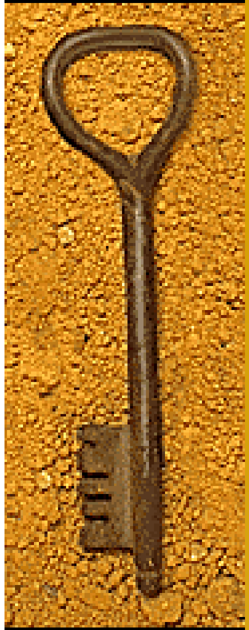
$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

HEINRICH HEINE
UNIVERSITÄT
DÜSSELDORF

Summary

- **SKP** provides us with a calculus that fits the mark in several respects:
  - it is extensional with respect to the standard connectives,
  - it contains an entailment connective satisfying many conditions we basically associate with a conditional,
  - its semantics is only minimally deviant, given the standard possible worlds approach.
  - given this semantics the logic is correct, so in light of the dialetheist's case stated in Chap. 2, we have – besides **LPQ** – a second logic that can serve is sound derivations (and even one fitting the criteria better).
- On the other hand, using a sequential calculus like that is not usual and one needs to accustom to it a bit. That, of course, is only a pragmatic concern.
- A more interesting question is what the intuitive reading of the SKP-conditional is, since it is not semantic entailment in the usual sense of alethic modal logic, nor is it Relevant entailment as in **R**. The accessibility relation used in **SKP** cannot be used for necessity, since the failure of reflexivity makes $\Box A \supset A$ false!

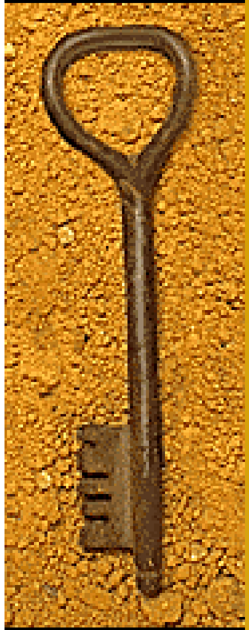


$p \wedge \neg p$

Manuel Bremer
Centre for Logic, Language and Information

Questions

- (Q1) In the truth conditions for "||-" we use an evaluation relation v and write $\neg v(B,1)$ and not $v(B,0)$ – why? And why can we *then* write just $\neg((A_1 \ldots A_n) \Vdash B)$ for $v((A_1 \ldots A_n) \Vdash B, 0)$?
- (Q2) Given (SKP→) explain the cases that an entailment is true, simply true, false, simply false, antinomic! The entailment being antinomic derives from which sentence being antinomic?
- (Q3) What clauses had to be added in (SKP→) and the tree rules for "→" to make Contraposition valid?



p ∧¬p

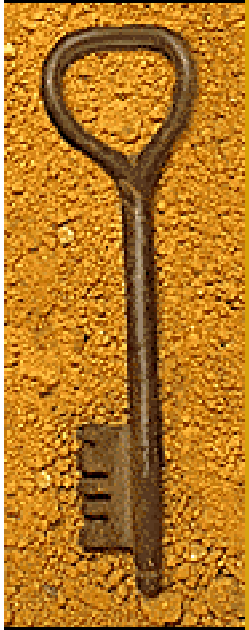
Manuel Bremer
Centre for Logic,
Language and
Information

Exercises

- (Ex1) Gives the definitions of a well-formed formula and a derivation for **SKP**.
- (Ex2) Proof in **SKP** the following theorems:
  $p \lor \neg p \rightarrow q \; ||\text{-} \; q$
  $\neg p \rightarrow q, p \rightarrow q \; ||\text{-} \; q$
  $p, q \; ||\text{-} \; p$ [like Weakening]
  $p \land q \; ||\text{-} \; p \lor q$
  $p \land q \; ||\text{-} \; q \land p$ [and the same for " ∨"]
  $(\forall x)F(x) \; ||\text{-} \; (\exists x)F(x)$
  $(\exists x)(\forall y)R(x,y) \; ||\text{-} \; (\forall y)(\exists x)R(x,y)$
  $(\exists x)(P(x) \rightarrow B), (\forall x)P(x) \; ||\text{-}B$ ["x" not in B]
  $(\exists x)(F(x) \rightarrow G(x)), (\forall x)F(x) \; ||\text{-} \; (\exists x)G(x)$
- (Ex3) Proof in **SKP**$^{+}$ the following rules:
  $\Gamma \; ||\text{-} \; A \rightarrow (\neg B \land B) \Rightarrow \Gamma \; ||\text{-} \; (\neg A)$
  $\Gamma \; ||\text{-} \; A \rightarrow B \; \& \; \Pi \; ||\text{-}(\neg B) \Rightarrow \Gamma \cup \Pi \; ||\text{-}(\neg A)$
  So indirect proofs and *Modus Tollens* are valid in **SKP**$^{+}$ but not in **SKP** itself.

Further Reading

- The basic paper on **SKP** is Priest, Graham. "Sense, Entailment and *Modus Ponens*", *Journal of Philosophical Logic*, 9 (1980), pp.415-35. The paper contains a second (algebraic) semantics, which, however, has less if any intuitive force compared to the possible worlds approach [see also (Bremer 1998, pp.100-110)].
- The classic paper on sequential calculi is (Gentzen 1934/35).
- On Natural Deduction as descendant of sequential calculi, and a brief presentation of the Gentzen approach, see (Essler/Martinez 1991).



$p \land \neg p$

Manuel Bremer
Centre for Logic, Language and Information